LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2020a, Prepared by Dr. Robert Gardner

February 7, 2020

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.**

You may omit two problems. Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit.

1. Find the solution set of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. (**A1, A7, B4**)

2. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. **(A9, C7)**

3. Find the projection of $[1, 2, 3, 4]$ onto the plane $x+2y+z-w = 0$. Explain your reasoning! **(B3, B7, B8, C17, C19)**

4. State the definition of *vector space.* **(C1)**

5. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$, where $\mathcal{P}_3$ is the vector space of all polynomials of degree 3 or less, be defined by $T((p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. **(C7, C8, C11, C15)**

6. Consider the vectors $\vec{v}_1 = x^2 - 2x + 1$, $\vec{v}_2 = 2x^2 + 5x + 11$, and $\vec{v}_3 = 3x^2 + 7x + 17$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. **(C5, C11, C15)**

7. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \le \|\vec{v}\|\|\vec{w}\|$, to prove the Triangle Inequality in an inner-product space. **(B8, B10, C15)**

8. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

9. If A is an invertible $n \times n$ matrix, then $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$. Use this fact to find A^{-1} for

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}.$$

(D13, D15, D16)

10. Diagonalize $A = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix}$ and calculate A^{100}. Of course, you may leave your answer in terms of powers of certain numbers. (**D1, D17, D20**)