LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2015a, Prepared by Dr. Robert Gardner
February 13, 2015

NAME __________________________ Start Time: ______ End Time: ______

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side**. You may omit two problems. Indicate which two problems you are omitting: _____ and ____. There is a three hour time limit.

1. Consider the matrix $A = \begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}$. Put $A$ in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. (A3, A4, A5)

2. Solve the system of equations (A2, A6, A7):
   
   \begin{align*}
   x_2 & - 3x_3 = -5 \\
   2x_1 & + 3x_2 - x_3 = 7 \\
   4x_1 & + 5x_2 - 2x_3 = 10
   \end{align*}

3. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. (A9, C7)

4. State the definition of vector space. (C1)

5. Let $T : P_3 \to P_3$ be defined by $T(p(x))$, the derivative of $p(x)$. let the ordered basis for $P_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix $A$ which represents $T$ relative to $B, B'$. (C7, C8, C11, C15)

6. Find the projection of $x$ onto sin $x$ in the inner product space $C_{0,\pi}$ of continuous functions on $[0, \pi]$ with the inner product of $f$ and $g$ defined as (B8, C15, C17):
   
   $$\langle f, g \rangle = \int_0^\pi f(x)g(x) \, dx.$$

7. Use the Schwarz Inequality, which states that for vectors $\vec{v}$ and $\vec{w}$ in an inner-product space, we have $|\langle \vec{v}, \vec{w} \rangle| \leq ||\vec{v}|| ||\vec{w}||$, to prove the Triangle Inequality in an inner-product space. (B8, B10, C15)
8. Use row reduction and the properties of determinants to calculate the determinant of

\[
A = \begin{bmatrix}
2 & 1 & 3 & 4 \\
6 & 2 & 1 & 4 \\
6 & 3 & 9 & 12 \\
2 & 1 & 3 & 4 \\
\end{bmatrix}.
\]

Explain your reasoning. (D12, D14, D15)

9. Diagonalize \( A = \begin{bmatrix}
1 & -3 & 3 \\
0 & -5 & 6 \\
0 & -3 & 4 \\
\end{bmatrix} \). (D2, D17, D18, D19, D20)

10. Find the L/U decomposition of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -1 \\
2 & 8 & 4 \\
-1 & 3 & 4 \\
\end{bmatrix}.
\]

Explain your reasoning. (D23)