LINEAR ALGEBRA COMPREHENSIVE EXAM
Spring 2018a, Prepared by Dr. Robert Gardner
January 26, 2018

NAME ___________________________ Start Time: _______ End Time: _______

Be clear and give all details. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators and turn off your cell phones! Use the paper provided and only write on one side.

You may omit two problems. Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit.

1. Find the solution of \( A\vec{x} = \vec{b} \) where

\[
A = \begin{bmatrix}
1 & -4 & 1 \\
3 & -13 & 0 \\
2 & -9 & -1
\end{bmatrix}
\quad \text{and} \quad \vec{b} = \begin{bmatrix}
2 \\
-6 \\
-8
\end{bmatrix}
\]

and express the solution as a translation of a vector space. (A1, A7, B4)

2. Let \( A \) be an \( m \times n \) matrix. Show that \( \{\vec{x} \mid A\vec{x} = \vec{0}\} \) is a subspace of \( \mathbb{R}^n \). (A8, A9, C4)

3. Consider the inner product space \( C_{-\pi,\pi} \) of continuous functions on \([-\pi, \pi]\) with the inner product of \( f \) and \( g \) defined as

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx.
\]

In the vector space \( C_{-\pi,\pi} \) find the angle between \( \cos x \) and \( \sin x \). (B8, B9, C15)

4. State the definition of vector space. (C1)

5. Consider the vectors \( \vec{v}_1 = x^2 - 2x + 1, \vec{v}_2 = 2x^2 + 5x + 11, \) and \( \vec{v}_3 = 3x^2 + 7x + 17 \) in \( \mathcal{P}_2 \), the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using linear algebra and ordered bases. If you are not using ordered bases then you are not arguing correctly!!! (C5, C11, C15)

6. Find the standard matrix representation (i.e. the representation with respect to the standard bases of \( \mathbb{R}^m \) and \( \mathbb{R}^n \)) of the linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) defined by \( T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3) \). (C7, C8)
7. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\} of \(\mathbb{R}^4\). (C17, C19, C20, C21)

8. Show that the vector space

\[ V = \{m + a\cos(x) + b\sin(x) \mid m, a, b \in \mathbb{R}\} \]

is isomorphic to \(\mathbb{R}^3\) (actually construct the isomorphism). (C12, C13, C15)

9. Prove that if \(\lambda\) is an eigenvalue of an \(n \times n\) matrix \(A\), then the set \(E_{\lambda}\) consisting of the zero vector together with all eigenvectors of \(A\) for this eigenvalue \(\lambda\) is a subspace of \(n\)-space. (C4, D17, D19)

10. Find the L/U decomposition of the matrix

\[
A = \begin{bmatrix}
1 & 3 & -1 \\
2 & 8 & 4 \\
-1 & 3 & 4 \\
\end{bmatrix}.
\]

Explain your reasoning. (D23)