LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2016a, Prepared by Dr. Robert Gardner

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NAME ______________________________ Start Time: ________ End Time: ________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Find the solution of $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 3 & -13 & 0 \\ 2 & -9 & -1 \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} -2 \\ -10 \\ -8 \end{bmatrix}$$

and express the solution as a translation of a vector space. **(A1, A7, B4)**

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Give two conditions which would (each) imply that $A\vec{x} = \vec{b}$ has multiple solutions. **(A5, A8, A9)**

3. Show that the vectors $\sin x$ and $\cos x$ are orthogonal in the inner product space $C([0, 2\pi])$ of continuous functions on $[0, 2\pi]$ with inner product of f and g defined as

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$

(B8, B9, C15)

4. State the definition of *vector space*. **(C1)**

5. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain using ordered bases. **(C5, C11, C15)**

6. Find the orthogonal complement of $\mathrm{span}\{[-1, 2, 0, 3], [0, 4, 1, -2]\}$ in $\mathbb{R}^4$. **(C3, C18)**

7. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism and verify that it is an isomorphism). **(C12, C13, C15)**

8. Use the Gram-Schmidt process to find an orthonormal basis for the subspace $\text{span}\{[1,2,0,2], [2,1,1,1], [1,0,1,1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

9. Find the eigenvalues (they are integers) and the corresponding eigenvectors of:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

(A3, D14, D17, D18, D19)

10. Find the L/U decomposition of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 8 & 4 \\ -1 & 3 & 4 \end{bmatrix}.$$

Explain your reasoning. **(D23)**