LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2009, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Express the solution of this system as a translation of a vector space:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & & & = & 5 \\ -5x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & -7 \\ 5x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \end{array}$$

(A1, A2, A3, A7, B4)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 14 & -4 & 7 \\ -3 & -6 & 0 & -9 \\ 5 & -8 & 6 & 9 \\ 2 & 13 & -3 & 9 \end{bmatrix}.$$

Which columns contain pivots? What is a basis for the column space? **(A3, A4, A5, D6)**

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1,2,3]$ and $[4,5,6]$ and passes through the point $(7,8,9)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

4. State the "Fundamental Theorem of Finite Dimensional Vector Spaces" (which deals with isomorphisms of vector spaces). Let V_1 and V_2 be vector spaces with real scalars. What conditions must be satisfied for $\pi : V_1 \to V_2$ to be an isomorphism? Are the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ isomorphic? Explain. (**B4, B12**)

5. Let A be an $n \times n$ matrix. Prove that the collection of all solutions to the equation $A\vec{x} = \vec{0}$ form a subspace of $\mathbb{R}^n$. (**A9, C2, C4**)

6. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly dependent? Explain. **(C5, C11, C15)**

7. Use the Gram-Schmidt process to find an orthonormal basis for the subspace span$\{[1, 2, 0, 2], [2, 1, 1, 1], [1, 0, 1, 1]\}$ of $\mathbb{R}^4$. (**C17, C19, C20, C21**)

8. (a) What is an elementary matrix? **(D7)**

(b) Express A and A^{-1} as a product of elementary matrices where $A = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$. **(D3, D7, D8, D9)**

9. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)

10. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. What is the eigenspace? **(D12, D17, D18, D19)**