LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2005, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER _______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit two problems. Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. *Prove* that a linear system of equations with two different solutions has an infinite number of solutions. **(A1, A6, C7)**

2. Let A be an $m \times n$ matrix. Show that $\{\vec{x} \mid A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$. **(A8, A9, C4)**

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1, 2, 3]$ and $[4, 5, 6]$ and passes through the point $(7, 8, 9)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

4. State the definition of *vector space*. **(C1)**

5. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. **(C7, C8, C11, C15)**

6. Transform the basis $\{[1, 1, 0], [0, 1, 2], [1, 1, 1]\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. **(C17, C19, C20, C21)**

7. Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Show that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W. **(C4, C18, C19)**

8. Consider

$$A = \begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix}.$$

Find the rank, a basis for the row space, and a basis for the column space. (**A4, A5, D6, D10**)

9. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

10. What does it mean for matrix A to be an *oprthogonal matrix*? Prove that for $n \times n$ orthogonal matrix A, we have $\|A\vec{x}\| = \|A^{-1}\vec{x}\|$.