LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2004, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER _______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: _____ and _____. NO CALCULATORS!!! Time limit: 3 hours.

1. Express the solution of this system as a translation of a vector space:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & & & = & 5 \\ -5x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & -7 \\ 5x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \end{array}$$

(A1, A2, A3, A7, B4)

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 14 & -4 & 7 \\ -3 & -6 & 0 & -9 \\ 5 & -8 & 6 & 9 \\ 2 & 13 & -3 & 9 \end{bmatrix}.$$

Which columns contain pivots? What is a basis for the column space? **(A3, A4, A5, D6)**

3. Consider the plane in $\mathbb{R}^3$ which contains the vectors $[1, 2, 3]$ and $[4, 5, 6]$ and passes through the point $(7, 8, 9)$. Find the equation of the plane (in terms of x, y, and z coordinates) and express the plane as a translation of a vector space. **(B4, B12)**

4. Consider the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

In the vector space $C_{-\pi,\pi}$, find the angle between $\cos x$ and $\sin x$. **(B8, B9, C15)**

5. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism and verify that it is an isomorphism). **(C12, C13, C15)**

6. Consider the vectors $\vec{v}_1 = x^2 + 2x + 3$, $\vec{v}_2 = 7x^2 - 5x + 2$, and $\vec{v}_3 = -4x^2 + 2x - 9$ in $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less. Are these vectors linearly independent? Explain. **(C5, C11, C15)**

7. Find the orthogonal complement of $\text{span}\{[-1, 2, 0, 3], [0, 4, 1, -2]\}$ in $\mathbb{R}^4$. **(C3, C18)**

8. Let A and C be matrices such that the product AC is defined. Prove that the column space of AC is contained in the column space of A. **(D6, D10)**

9. Find the eigenvalues (they are integers) and the eigenvectors of:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

(A9, D14, D17, D18, D19)

10. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of A. **(A4, D15)**