LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2003, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Express the solution of this system as a translation of a vector space:

$$\begin{array}{rcrcrcrcr} 3x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\ 5x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & = & 14 \\ -5x_1 & - & 3x_2 & - & x_3 & + & x_4 & = & -8. \end{array}$$

   (**A1, A2, A3, A7, B4**)

2. Give three conditions on $n \times n$ matrix A which would (each) imply that the system $A\vec{x} = \vec{b}$ has a unique solution. Does the system

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -2 & 5 & 7 \\ 1 & 11 & 16 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 31 \\ 70 \end{bmatrix}$$

   have a unique solution (explain)? (**A5, A8, A9**)

3. Consider the inner product space $C_{-\pi,\pi}$ of continuous functions on $[-\pi, \pi]$ with the inner product of f and g defined as

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

   One can show that

$$\left| \int_{-\pi}^{\pi} f(x)g(x)\, dx \right| \leq \sqrt{\int_{-\pi}^{\pi} (f(x))^2\, dx} \sqrt{\int_{-\pi}^{\pi} (g(x))^2\, dx}.$$

   Use this fact (which is Schwarz's Inequality in $C_{-\pi,\pi}$) to prove the triangle inequality in this space. (**B8, B10, C15**)

4. In the vector space $C_{-\pi,\pi}$ of problem 3, find the angle between $\cos x$ and $\sin x$. (**B8, B9, C15**)

5. State the definition of *vector space*. (**C1**)

6. Find the standard matrix representation (i.e. the representation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$) of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3)$. (**C7, C8**)

7. Consider the space $\mathcal{P}_3$ of all polynomials of degree 3 or less. Find the coordinate vector of $x^3 + 3x^2 - 4x + 2$ relative to the ordered basis$(x, x^2 - 1, x^3, 2x^2)$. (**C11, C6, A1**).

8. Transform the basis $\{[1, 1, 1], [1, 0, 1], [0, 1, 1]\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

9. Express A and A^{-1} as products of elementary matrices where

$$A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}.$$

(**D3, D7, D8, D9**)

10. Let A and C be matrices such that the product AC is defined. Prove that the column space of AC is contained in the column space of A. (**D6, D10**)

11. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

12. Consider

$$A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.$$

Put A in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of A. **(A4, D15)**