NAME ___________________________ STUDENT NUMBER ___________________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators!!! You may omit two numbered problems. Indicate which two problems you are omitting: _____ and _____.

1. Express the solution of this system as a translation of a vector space:

\[
\begin{align*}
3x_1 + 2x_2 + x_3 &= 6 \\
x_1 + x_2 + x_3 + x_4 &= 4 \\
5x_1 + 4x_2 + 3x_3 + 2x_4 &= 14 \\
-5x_1 - 3x_2 - x_3 + x_4 &= -8.
\end{align*}
\]

(A1, A2, A3, A7, B4)

2. Give three conditions on \( n \times n \) matrix \( A \) which would (each) imply that the system \( A\vec{x} = \vec{b} \) has a unique solution. Does the system

\[
A = \begin{bmatrix}
1 & 2 & 2 \\
-2 & 5 & 7 \\
1 & 11 & 16
\end{bmatrix}
\quad \vec{x} = \begin{bmatrix}
10 \\
31 \\
70
\end{bmatrix}
\]

have a unique solution (explain)? (A5, A8, A9)

3. Consider the inner product space \( C_{-\pi, \pi} \) of continuous functions on \([-\pi, \pi]\) with the inner product of \( f \) and \( g \) defined as

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx.
\]

One can show that

\[
\left| \int_{-\pi}^{\pi} f(x)g(x) \, dx \right| \leq \sqrt{\int_{-\pi}^{\pi} (f(x))^2 \, dx} \sqrt{\int_{-\pi}^{\pi} (g(x))^2 \, dx}.
\]

Use this fact (which is Schwarz’s Inequality in \( C_{-\pi, \pi} \)) to prove the triangle inequality in this space. (B8, B10, C15)

4. In the vector space \( C_{-\pi, \pi} \) of problem 3, find the angle between \( \cos x \) and \( \sin x \). (B8, B9, C15)

5. State the definition of vector space. (C1)
6. Find the standard matrix representation (i.e. the representation with respect to the standard bases of \( \mathbb{R}^m \) and \( \mathbb{R}^n \)) of the linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^4 \) defined by \( T((x_1, x_2, x_3)) = (x_1 + x_2, x_2 + x_3, x_3 + x_1, x_1 + x_2 + x_3) \). (C7, C8)

7. Consider the space \( \mathcal{P}_3 \) of all polynomials of degree 3 or less. Find the coordinate vector of \( x^3 + 3x^2 - 4x + 2 \) relative to the ordered basis \((x, x^2 - 1, x^3, 2x^2)\). (C11, C6, A1)

8. Transform the basis \([1,1,1], [1,0,1], [0,1,1] \) for \( \mathbb{R}^3 \) into an orthogonal basis using the Gram-Schmidt process. (C17, C19, C20, C21)

9. Express \( A \) and \( A^{-1} \) as products of elementary matrices where

\[
A = \begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}.
\]

(D3, D7, D8, D9)

10. Let \( A \) and \( C \) be matrices such that the product \( AC \) is defined. Prove that the column space of \( AC \) is contained in the column space of \( A \). (D6, D10)

11. Find the eigenvalues (they are integers) and the eigenvectors of (A9, D14, D17, D18, D19):

\[
A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.
\]

12. Consider

\[
A = \begin{bmatrix} 2 & 2 & 0 & 4 \\ 3 & 3 & 2 & 2 \\ 0 & 1 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix}.
\]

Put \( A \) in a row echelon form, keeping track of how each row operation affects the determinant. Then calculate the determinant of \( A \). (A4, D15)