LINEAR ALGEBRA COMPREHENSIVE EXAM

Spring 2002, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Consider the matrix $A = \begin{bmatrix} 0 & 2 & -1 & 3 \\ -1 & 1 & 2 & 0 \\ 1 & 1 & -3 & 3 \\ 1 & 5 & 5 & 9 \end{bmatrix}$. Put A in (1) row echelon form, and (2) reduced row echelon form. At each step, tell the elementary row operation you are using. (**A3, A4, A5**)

2. Solve the system of equations (**A2, A6, A7**):

$$\begin{array}{rcrcrcr} & & x_2 & - & 3x_3 & = & -5 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 7 \\ 4x_1 & + & 5x_2 & - & 2x_3 & = & 10 \end{array}$$

3. Prove that if $\vec{x}_1$ and $\vec{x}_2$ are both solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, then any linear combination of $\vec{x}_1$ and $\vec{x}_2$ is also a solution. (**A9, C7**)

4. Explain the difference between a *vector* in $\mathbb{R}^n$ and a *point* in $\mathbb{R}^n$. (**B1, B3**)

5. Use Schwarz's Inequality, which states that for $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ we have $|\vec{v}_1 \cdot \vec{v}_2| \leq \|\vec{v}_1\|\|\vec{v}_2\|$ to prove the Triangle Inequality in $\mathbb{R}^n$. (**B8, B10**)

6. Show that $\mathcal{P}_2$, the vector space of all polynomials of degree 2 or less, is isomorphic to $\mathbb{R}^3$ (actually construct the isomorphism). (**C12, C13, C15**)

7. Find the projection of x onto $\sin x$ in the inner product space $C_{0,\pi}$ of continuous functions on $[0, \pi]$ with the inner product of f and g defined as (**C15, C17**)

$$\langle f, g \rangle = \int_0^\pi f(x)g(x)\, dx.$$

8. Let $T : \mathcal{P}_3 \to \mathcal{P}_3$ be defined by $T(p(x)) = D(p(x))$, the derivative of $p(x)$. Let the ordered basis for $\mathcal{P}_3$ be $B = B' = (x^3, x^2, x, 1)$. Find the matrix A which represents T relative to B, B'. (**C7, C8, C11, C15**)

9. Transform the basis $\{[1, 1, 1], [1, 0, 1], [0, 1, 1]\}$ for $\mathbb{R}^3$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

10. Find the rank, nullity, and a basis for the column space of (**A3, 4, A5, D6, D10**):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -4 & 2 \\ 3 & -10 & -7 \end{bmatrix}.$$

11. Find the eigenvalues (they are integers) and the eigenvectors of (**A9, D14, D17, D18, D19**):

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -5 & -2 & -5 \\ 5 & 0 & 3 \end{bmatrix}.$$

12. Prove that if λ is an eigenvalue of an $n \times n$ matrix A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space. (**C4, D17, D19**)