CALCULUS COMPREHENSIVE EXAM

Spring 2011, Prepared by Dr. Jeff Knisley
March 25, 2011

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. (a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).
   (b) Use the definition to prove that if $\lim_{x\to p} f(x) = L$ and $k \in \mathbb{R}$, then $\lim_{x\to p} [kf(x)] = kL$. (**1**)

2. Prove that if f is differentiable at $x = p$, then f is also continuous at $x = p$. (**4,7**)

3. (a) State the definition of a function $f(x)$ being continuous at $x = p$.
   (b) Show that the following function is continuous at $x = 0$. (**4, 33**)

$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

4. Consider the function

$$f(x) = \frac{x^4}{1 - x^4}$$

   Find the first and second derivative of f, determine where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$ **(3, 8, 14, 15, 16, 17)**

5. (a) State both parts of the Fundamental Theorem of Calculus.
   (b) Calculate $F'(x)$ and simplify completely given that $F(x) = \int_0^{\sec(x)} \frac{1}{\sqrt{t^2 - 1}} dt$.

Indicate with a star $(*)$ where you have used the Fundamental Theorem of Calculus in your computation. **(20, 23, 34)**

6. What is the length of the curve $y = \cosh(x)$ for x in $[0, \ln(2)]$? **(23, 24, 27)**

7. Evaluate (**37, 38, 39**)
(a) $\lim_{x \to 0^+} \ln(x^x)$
(b) $\int_{-\infty}^{\infty} \frac{|x|}{1+x^4} dx$

8. Prove the following using the definition of the limit of a sequence: If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of real numbers with $0 \leq a_n \leq b_n$ for all $n \in \mathbb{Z}^+$ and if
$$\lim_{n \to \infty} b_n = 0$$
then the limit of $\{a_n\}_{n=1}^{\infty}$ exists and is also 0. **(41)**

9. Do each of the following (**46**)**:**
(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e., on what types of sets might the series converge conditionally, converge absolutely, or diverge).
(b) What is the radius of converge of the series
$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$$
(give detailed reasons for your answer).

10. Do each of the following (**44, 47**):
(a) Use the MacLaurin Series for $f(x) = \sin(x)$ to find the Maclaurin series expansion of $\int_0^x \sin(u^4)\ du$.
(b) Use the series in (a) to calculate the limit
$$\lim_{x \to 0} \frac{1}{x^5} \int_0^x \sin\left(u^4\right) du$$