CALCULUS COMPREHENSIVE EXAM

Spring 2010, Prepared by Dr. Jeff Knisley

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NAME ______________________________ STUDENT NUMBER _______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Do each of the following:

(a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).

(b) Prove that if $\lim_{x\to a} f(x) = L$ and $k \neq 0$, then $\lim_{x\to a}(kf(x)) = kL$. **(1)**

2. Do each of the following **(3, 33)**:

(a) State the Sandwich Theorem (sometimes called the "Squeeze Theorem") for the limit of a function.

(b) Use the fact that $\sin\theta < \theta < \tan\theta$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ (WARNING: This is a two sided limit and the inequality is only known to hold for $\theta \in (0, \pi/2)$.)

3. Consider $f(x) = \dfrac{x^3}{x(1-x^2)}$. Find the first and second derivative of f, find where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$. **(8, 14, 15, 16, 17)**

4. For which two positive numbers whose sum is 10 is the sum of their squares a maximum?

5. Do each of the following**(23, 24, 35)**:

(a) State the two parts of the Fundamental Theorem of Calculus.

(b) Use the Fundamental Theorem of Calculus to evaluate $\int_0^{\pi/2} x\sin x\, dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

6. Find the length of $y = \cosh(x)$ for $x \in [0, 1]$ **(23)**.

7. Evaluate **(37, 38, 39)**:

(a) $\lim_{x \to 0^+} x^x$.

(b) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ (use all notation correctly and don't write things that don't make sense).

(c) Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2}\, dx$.

8. Do each of the following **(41, 43)**:

(b) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\lim_{n \to \infty} a_n = L$."

(c) Use the Integral Test to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. Do each of the following **(46)**:

(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge?).

(b) What is the radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}$ (give detailed reasons for your answer).

10. Do each of the following **(44, 47)**:

(a) Use the MacLaurin series for e^x to find a series for $\int e^{-x^2}\, dx$.

(b) Estimate $\int_0^1 e^{-x^2}\, dx$ to the nearest 0.001 and explain why you know your answer has this level of accuracy.