CALCULUS COMPREHENSIVE EXAM

Fall, 2010, Prepared by Dr. Jeff Knisley

October 15, 2010

NAME _____ STUDENT NUMBER _____

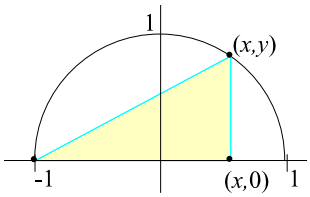
Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. (a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).
   (b) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M$. **(1, 2)**

2. Do each of the following **(5)**:
   (a) State the Intermediate Value Theorem.
   (b) Prove that $3^x = x^4$ for some x.

3. Prove that if f has a derivative at $x = c$, then f is continuous at $x = c$. **(4,7)**

4. A triangle has vertices at $(-1, 0)$, $(x, 0)$, and (x, y), where $x^2 + y^2 = 1$ and $y \geq 0$ (see below). For what point (x, y) on the unit circle is the area of the triangle a maximum?



5. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x)\,dx$.
   (b) Explain the difference between a definite integral and an indefinite integral (if any). (**20, 21, 23**)

6. Do each of the following:
   (a) State the Fundamental Theorem of Calculus (both parts). **(23)**
   (b) Use the Fundamental Theorem of Calculus to evaluate

$$\int_0^1 \frac{1 - x^2}{1 + x^2}\,dx$$

and indicate with a star $(*)$ where you are applying the Fundamental Theorem. **(24, 35)**

7. Do each of the following.
   (a) Evaluate $\lim_{x\to 0^+} (1+x)^{1/x}$. **(31, 37)**
   (b) Evaluate $\int_{-\infty}^{\infty} \frac{dx}{|x|+1}$. **(24, 38)**
   (c) Evaluate $\int_{-1}^{1} \frac{1}{x^2}\,dx$. **(39)**

8. Do each of the following **(41, 43)**:
   (a) State the definition of the limit of a sequence: $\lim_{n\to\infty} a_n = L$.
   (b) State the definition of the sum of a series: $\sum_{n=1}^{\infty} a_n = S$.
   (c) Use the Integral Test to show that a p-series with $0 < p < 1$ diverges.

9. Consider $\sum_{n=1}^{\infty} \frac{n+1}{n!}\left(\frac{x}{2}\right)^n$. Find the interval of convergence, radius of convergence, and values of x for which the convergence is absolute or conditional. (**44, 45, 46**)

10. (a) Find a power series for $f(x) = x^2 e^{-x^2}$.
    (b) Use your power series to calculate $\int_0^1 x^2 e^{-x^2}\,dx$ (as a series). How many nonzero terms of the series representation of this integral must be summed to approximate the integral to the nearest $\frac{1}{100}$? (**44, 47**)