CALCULUS COMPREHENSIVE EXAM

Fall 2007, Prepared by Dr. Robert Gardner

December 7, 2007

NAME _____ STUDENT NUMBER _____

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 though 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit. No calculutors (this is a *math* test)!

1. (a) State the definition of the limit of a function (i.e., what does $\lim_{x\to a} f(x) = L$ mean?). **(1)**

(b) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M$ **(1,2)**

2. Do each of the following:

(a) State the Sandwich Theorem (also called the Squeeze Theorem) for the limit of a function. **(3)**

(b) Use the fact that $\sin(\theta) < \theta < \tan(\theta)$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} = 1$. WARNNING: This is a two sided limit and the inequality is known to hold only for $\theta \in (0, \pi/2)$. **(3)**

3. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$ and the volume of a sphere by $V = \frac{4}{3}\pi r^3$. **(8, 12, 16, 18)**

4. (a) State the Fundamental Theorem of Calculus (both parts). **(23)**

(b) Use the Fundamental Theorem of Calculus to evaluate $\int_0^1 x \sin x dx$ and indicate with a star (*) where you are applying the Fundamental Theorem. **(23, 24)**

5. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\displaystyle\int_a^b f(x)\, dx$. **(21)**

(b) Explain the difference between a definite integral and an indefinite integral (if any). **(20, 23)**

6. Consider $y = \dfrac{x^2+1}{e^x}$. Where is y increasing/decreasing? Where is y concave up/concave down? What are the asymptotes of y? Graph. **(14,15)**

7. The curve $y = \ln x$ for $x \in [1, e]$ is revolved about the y-axis to produce a water tank. How much work is done in pumping the tank full of water? The water starts at a level of $y = 0$, distances are measured in feet and the mass-density of water is 62.4 lb/ft^3. **(24,27,30)**

8. Do each of the following: **(33,34,35)**

(a) Evaluate $\lim\limits_{y\to 0} \dfrac{\sin 7y}{4y}$.

(b) Evaluate $\lim\limits_{t\to 0^+} \left(1+\dfrac{1}{t}\right)^t$.

(c) Evaluate $\displaystyle\int \frac{\sec^2 x\, dx}{\sqrt{1-\tan^2 x}}$.

9. State the Integral Test (which concerns the convergence of a positive term series). Show that for $p > 1$ the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. **(38, 43)**

10. Do each of the following:

(a) Use the MacLaurin Series for e^x to find a series for $\displaystyle\int e^{-x^2} dx$.
(30, 46)

(b) Estimate $\displaystyle\int_0^1 e^{-x^2} dx$ to the nearest 0.001 and explain why you know your answer has this level of accuracy. **(44, 47)**