CALCULUS COMPREHENSIVE EXAM

Fall 2004, Prepared by Dr. Robert Gardner

December 3, 2004

NAME ________________________ STUDENT NUMBER ____________________________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 though 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit.

1. Do each of the following **(1)**:

(a) State the definition of the limit of a function (i.e., what does $\lim_{x\to a} f(x) = L$ mean?).

(b) Use the definition of limit to prove that $\lim_{x\to 8}\left(\frac{x}{2}+5\right) = 9$.

2. Do each of the following **(5)**:

(a) State the Intermediate Value Theorem.

(b) Prove that $\cos x = x$ for some x.

3. Do each of the following **(10)**:

(a) What does it mean for $y = f(x)$ to be a function *implicit* to the equation $F(x, y) = 0$?

(b) Find the equation of the line tangent to $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

4. Consider $f(x) = \dfrac{x^2}{x^2-1}$. Do each of the following **(15, 16, 17)**:

(a) On what intervals is f concave up and on what intervals is f concave down?

(b) What are the asymptotes of f?

(c) Graph f.

5. Do each of the following **(23, 24)**:

(a) State the two parts of the Fundamental Theorem of Calculus.

(b) Use the Fundamental Theorem of Calculus to evaluate $\int_0^1 x\sin x\,dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

6. If a plant releases an amount A of pollutant into a canal at time $t = 0$, then the resulting concentration of pollutant at time t in the water at a town on the canal at distance x_0 from the plant is

$$C(t) = \frac{A}{\sqrt{k\pi t}} \exp\left(-\frac{x_0^2}{4kt}\right)$$

where k is a certain constant. Show that the maximum concentration at the town is

$$C_{max} = \frac{A}{x_0}\sqrt{\frac{2}{\pi e}}.$$

(18, 30, 31)

7. Do each of the following **(31, 37, 39)**:

(a) Evaluate $\displaystyle\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$.

(b) Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{1}{x^2}\,dx$.

8. Do each of the following **(39, 41, 43)**:

(a) If f is continuous on $[0,\infty)$, then state the definition of $\displaystyle\int_0^\infty f(x)\,dx$. You may assume the usual definition for integrals of continuous functions on closed and bounded intervals has been established.

(b) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\displaystyle\lim_{n\to\infty} a_n = L$."

(c) Use the Integral Test to show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

9. Do each of the following **(46)**:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e., on what types of sets might the series converge conditionally, converge absolutely, or diverge)?

(b) What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$?

10. Compute the Taylor series for e^{-x^2} and $\displaystyle\int_0^x e^{-t^2}\,dt$. **(47)**