CALCULUS COMPREHENSIVE EXAM

Fall 2002, Prepared by Dr. Robert Gardner

December 5, 2002

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. Do each of the following (**1**):

   a. State the definition of the limit of a function (that is, what does $\lim_{x \to a} f(x) = L$ mean?).

   b. Use the definition to prove that $\lim_{x \to a} mx + b = ma + b$.

2. Prove that if f has a derivative at $x = c$, then f is continuous at $x = c$. (**4, 7**)

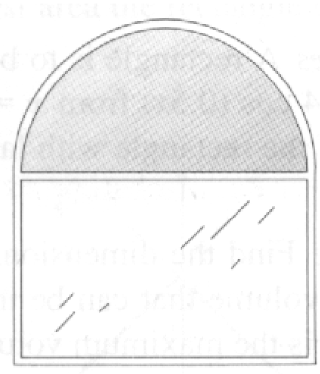
3. Consider $f(x) = \dfrac{x^3}{x(1-x)^2}$. Find the first and second derivatives of f, find where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$. (**8, 14, 15, 16, 17**)

4. Do each of the following (**10**):

   a. What does it mean for $y = f(x)$ to be a function *implicit* to the equation $F(x, y) = 0$?

   b. Find the equation of the line tangent to $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$.

5. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. (**18**)



6. Do each of the following (**23, 24, 31**):

   a. State the Fundamental Theorem of Calculus (both parts).

   b. Evaluate $\displaystyle\int_1^e \ln x\, dx$ (HINT: use parts) and indicate with a star $(*)$ where you have used the Fundamental Theorem of Calculus in your computations.

7. Find the length of $y = x^2$ for $x \in [0, 1]$. (**27, 34**)

8. The region bounded by the positive x-axis, the positive y-axis, and $y = e^{-x}$ is revolved about the y -axis. What's the volume? (**26, 31, 38**)

9. Do each of the following (**33, 34, 35**):

   a. Evaluate $\displaystyle\lim_{y\to 0} \frac{\sin 3y}{4y}$.

   b. Evaluate $\displaystyle\lim_{x\to 0^+} \left(1 + \frac{1}{x}\right)^x$.

   c. Evaluate $\displaystyle\int \frac{\sec^2 x\, dx}{\sqrt{1 - \tan^2 x}}$.

10. Do each of the following (**41**):

    a. Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\displaystyle\lim_{n\to\infty} (a_n) = L$."

    b. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series. Define *partial sum* of the series and define "$\displaystyle\sum_{n=1}^{\infty} a_n = L$."

11. Consider $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2 + 3}}$. Find the interval of convergence, the radius of convergence, and the values for which the convergence is absolute or conditional. (**44, 45, 46**)

12. Do each of the following (**44, 47**):

    a. Use the MacLaurin series for e^x to find a series for $\displaystyle\int e^{-x^2}\, dx$.

    b. Estimate $\displaystyle\int_0^1 e^{-x^2}\, dx$ to the nearest 0.001 and explain why you now your answer has this level of accuracy.