CALCULUS COMPREHENSIVE EXAM
Fall 2003, Prepared by Dr. Robert Gardner
November 21, 2003

NAME ___________________________ STUDENT NUMBER ________________________

Be clear and give all details. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators! You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and ____. There is a three hour time limit.

1. Do each of the following:
   (a) State the Sandwich Theorem (sometimes called the “Squeeze Theorem”) for the limit of a function.
   (b) Use the facts that \( \sin \theta < \theta < \tan \theta \) for \( \theta \in (0, \pi/2) \) to show that \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) (WARNING: This is a two sided limit and the inequality is only known to hold for \( \theta \in (0, \pi/2) \).) (3,33)

2. Do each of the following:
   (a) State the definition of derivative of a function \( f \). (6)
   (b) Use the definition to differentiate \( f(x) = \frac{1}{\sqrt{x}} \). (2, 6, 8)

3. Do each of the following:
   (a) State the Extreme Value Theorem. (12)
   (b) Show that the largest area rectangle is in fact a square. (12, 18)

4. Do each of the following:
   (a) State the definition of partition, norm of a partition, Riemann sum, and definite integral for \( \int_{a}^{b} f(x) \, dx \). (21)
   (b) Explain the difference between a definite integral and an indefinite integral (if any). (20, 23)
5. Do each of the following:
   (a) State the Fundamental Theorem of Calculus (both parts). (23)
   (b) Use the Fundamental Theorem of Calculus to evaluate $\int_{0}^{\ln 2} xe^x \, dx$ and indicate with a start (*) where you are applying the Fundamental Theorem. (24, 30)

6. Do each of the following:
   (a) State the definition of $\ln x$ (in terms of definite integrals). (29)
   (b) Use the definition to prove that $\ln ab = \ln a + \ln b$ ($a > 0$, $b > 0$) (23, 29)

7. Find the volume of the solid generated by revolving about the $x$-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$ and the line $x = 1$. (24, 26)

8. Do each of the following:
   (a) Evaluate $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$. (Show all work!) (30, 37)
   (b) Evaluate $\int_{0}^{2} \frac{dx}{(x - 1)^2}$. (39)
   (c) Use the Integral Test to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. Do each of the following:
   (a) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define “$\lim_{n \to \infty} a_n = L$.”
   (b) Let $\sum_{n=1}^{\infty} a_n$ be a series. Define partial sum of the series and define “$\left(\sum_{n=1}^{\infty} a_n\right) = L$. “ (41)

10. Find a Maclaurin Series for $f(x) = e^x$ (show your work). Where does the series converge absolutely? Where does it converge conditionally? Where does it diverge? Use the series to verify that $\int e^x \, dx = e^x + C$. (31, 45, 46, 47)