CALCULUS COMPREHENSIVE EXAM
Fall 2003, Prepared by Dr. Robert Gardner
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NAME ____________________________ STUDENT NUMBER ____________________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators!

1. (a) State the definition of the limit of a function (that is, what does \( \lim_{x \to a} f(x) = L \) mean?).
   (b) Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then \( \lim_{x \to a} (f(x) + g(x)) = L + M \). (1, 2)

2. Prove that if \( f \) has a derivative at \( x = c \), then \( f \) is continuous at \( x = c \). (4, 7)

3. (a) What does it mean for \( f(x) \) to be implicit to the equation \( F(x, y) = 0 \)?
   (b) Find \( y' \) if \( \sin(xy) = \ln(x\cot y) \). (8, 10, 31, 34)

4. Find the volume of the largest right circular cylinder which can be inscribed in a right circular cone of height 3 and base radius 1. The volume of a right circular cylinder of radius \( r \) and height \( h \) is \( V = \pi r^2 h \) and the volume of a right circular cone of height \( H \) and base radius \( R \) is \( V = \frac{1}{3}\pi R^2 H \). (18)

5. (a) State the Fundamental Theorem of Calculus (both parts).
   (b) Evaluate \( \int_1^e \ln x \, dx \) (HINT: use parts) and indicate with a star (*) where you have used the Fundamental Theorem of Calculus in your computations. (23, 24, 31)

6. Consider a cylindrical tank of height 10 ft and radius 4 ft. If the tank is full of water, find the work required to pump the water out of the top of the tank. The weight-density of water is 62.4 lb/ft³. Include units! (27)

7. (a) Use the definition of \( y = \tan^{-1} x \) (in terms of the tangent function) and implicit differentiation to find \( y' = \frac{d}{dx}[\tan^{-1} x] \).
   (b) Evaluate \( \int \frac{dx}{x^2 - 2x + 5} \). (28, 34, 35)

8. (a) If \( f \) is continuous on \( [a, c) \cup (c, b] \) then state the definition of \( \int_a^b f(x) \, dx \). That is, how do you integrate over a discontinuity? You may assume the usual definition for integrals of continuous functions has been established.
   (b) Evaluate \( \int_0^2 \frac{1}{(x - 1)^2} \, dx \).
   (c) Evaluate \( \lim_{x \to 0^+} x^x \). (37, 39)
9. State the Integral Test (which concerns the convergence of a positive term series). Show that for $p > 1$, the $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. (38, 43)

10. Find a MacLaurin Series for $f(x) = e^x$ (show your work). Where does the series converge absolutely? Where does it converge conditionally? Where does it diverge? Use the series to verify that $\int e^x \, dx = e^x + C$. 