CALCULUS COMPREHENSIVE EXAM

Fall 2003, Prepared by Dr. Robert Gardner

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NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit.

1. (a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).
   (b) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M$. **(1,2)**

2. Prove that if f has a derivative at $x = c$, then f is continuous at $x = c$. **(4,7)**

3. (a)What does it mean for $f(x)$ to be implicit to the equation $F(x, y) = 0$?
   (b) Find y' if $\sin(xy) = \ln(x \cot y)$. **(8, 10, 31, 34)**

4. Find the volume of the largest right circular cylinder which can be inscribed in a right circular cone of height 3 and base radius 1. The volume of a right circular cylinder of radius r and height h is $V = \pi r^2 h$ and the volume of a right circular cone of height H and base radius R is $V = \frac{1}{3}\pi R^2 H$. **(18)**

5. (a) State the Fundamental Theorem of Calculus (both parts).
   (b) Evaluate $\int_1^e \ln x\, dx$ (HINT: use parts) and indicate with a star $(*)$ where you have used the Fundamental Theorem of Calculus in your computations. **(23, 24, 31)**

6. Consider a cylindrical tank of height 10 ft and radius 4 ft. If the tank is full of water, find the work required to pump the water out of the top of the tank. The weight-density of water is 62.4 lb/ft^3. Include units! **(27)**

7. (a) Use the definition of $y = \tan^{-1} x$ (in terms of the tangent function) and implicit differentiation to find $y' = \frac{d}{dx}[\tan^{-1} x]$.
   (b) Evaluate $\int \frac{dx}{x^2 - 2x + 5}$. **(28, 34, 35)**

8. (a) If f is continuous on $[a, c) \cup (c, b]$ then state the definition of $\int_a^b f(x)\, dx$. That is, how do you integrate over a discontinuity? You may assume the usual definition for integrals of continuous functions has been established.
   (b) Evaluate $\int_0^2 \frac{1}{(x-1)^2}\, dx$.
   (c) Evaluate $\lim_{x\to 0^+} x^x$. **(37, 39)**

9. State the Integral Test (which concerns the convergence of a positive term series). Show that for $p > 1$, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. **(38, 43)**

10. Find a MacLaurin Series for $f(x) = e^x$ (show your work). Where does the series converge absolutely? Where does it converge conditionally? Where does it diverge? Use the series to verify that $\int e^x\,dx = e^x + C$.