CALCULUS COMPREHENSIVE EXAM

Fall 2002, Prepared by Dr. Robert Gardner

September 27, 2002

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

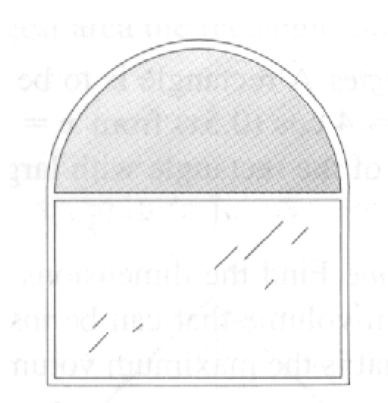
1. (a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).
   (b) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) + g(x)) = L + M$. (**1, 2**)

2. (a) State the Chain Rule (with all hypotheses).
   (b) Differentiate $y = \ln(\tan(e^{\sec x}))$. (**8, 31, 34**)

3. (a) State the definition of $f'(x)$, the derivative of $f(x)$.
   (b) Use the definition of derivative to find $f'(x)$ for $f(x) = \sqrt{x+2}$. (**2, 6**)

4. (a) Clearly state the relationship between the increasing-ness and decreasing-ness of a function and its first derivative.
   (b) Find the intervals on which $f(x) = x^3 - 12x - 5$ is increasing/decreasing and graph f. (**14**)

5. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. (**18**)



6. (a) State the Fundamental Theorem of Calculus (both parts).
   (b) Evaluate $\int_1^e \ln x\, dx$ (HINT: use parts) and indicate with a star $(*)$ where you have used the Fundamental Theorem of Calculus in your computations. (**23, 24, 31**)

7. The definition of $f(x) = \ln x$ is $f(x) = \displaystyle\int_1^x \frac{1}{t}\,dt$. The definition of e^x is $e^x = f^{-1}(x) = \ln^{-1}(x)$. Use these definitions to prove $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$ and $\dfrac{d}{dx}[e^x] = e^x$. (**23, 28, 29, 31**)

8. (a) State the definition of $y = \tan^{-1}(x)$.
   (b) Use this definition and implicit differentiation to find $y' = \dfrac{d}{dx}[\tan^{-1}(x)]$. (**35**)

9. (a) Evaluate $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. (Show all work!) (**30, 37**)
   (b) Evaluate $\displaystyle\int \frac{dx}{x^2+4x+8}$. (**24, 35**)
   (c) Evaluate $\displaystyle\int_0^2 \frac{dx}{(x-1)^2}$. (**39**)

10. (a) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\displaystyle\lim_{n\to\infty}(a_n) = L$."
    (b) Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series. Define *partial sum* of the series and define "$\displaystyle\left(\sum_{n=1}^{\infty} a_n\right) = L$." (**41**)

11. (a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)?
    (b) What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$? (**46**)

12. Compute a MacLaurin series for e^x and use it to verify that $\dfrac{d}{dx}[e^x] = e^x$. (**47**)