

CALCULUS COMPREHENSIVE EXAM

Summer 2013 Prepared by Dr. Robert Gardner

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NAME _____ Start time: _____ End time: _____

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Do each of the following:

- (a) State the definition of the limit of a function (i.e., what does $\lim_{x \rightarrow a} f(x) = L$ mean?). **(1)**
- (b) Use the definition from (a) to prove that if $\lim_{x \rightarrow a} f(x) = L$ and $c \in \mathbb{R}$, then $\lim_{x \rightarrow a} cf(x) = cL$. **(2)**

2. Do each of the following:

- (a) State the definition of *derivative* of a function f . **(6)**
- (b) Use the definition to differentiate $f(x) = \frac{1}{\sqrt{x}}$. **(2, 6, 8)**

3. Find the volume of the largest right circular cylinder which can be inscribed in a sphere of radius R . The volume of a right circular cylinder of radius r and height h is $V = \pi r^2 h$ and the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. **(12, 16, 18)**.

4. Do each of the following **(23, 24)**

- (a) State the two parts of the Fundamental Theorem of Calculus. **(23)**
- (b) Use the Fundamental Theorem of Calculus to evaluate $\int_0^e \ln x \, dx$ and indicate with a star (*) where you are applying the Fundamental Theorem. Hint: Integrate by parts. **(23, 24, 31)**

5. Do each of the following.

- (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x) \, dx$. **(21)**
- (b) Suppose the function $y = f(x)$ is implicit to the equation $x^2 + y^2 = 1$. Find y'' at $(\sqrt{2}/2, \sqrt{2}/2)$. **(10)**

6. Do each of the following:

(a) Use the definition of $y = \tan^{-1} x$ (in terms of the tangent function) and implicit differentiation to find $y' = \frac{d}{dx}[\tan^{-1} x]$. (10, 28, 35)

(b) Find the length of the curve given by the equation $y = \int_0^x \sqrt{\sec^4 t - 1} dt$ for $-\pi/4 \leq x \leq \pi/4$. (23, 27)

7. Do each of the following. Respect the calculus!

(a) Evaluate $\lim_{x \rightarrow 0^+} \left(1 + \frac{1}{x}\right)^x$. (31, 37)

(b) Evaluate $\int_{-1}^1 \frac{1}{x^2} dx$. (39)

8. Do each of the following:

(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? (46)

(b) What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$? (45)

9. Do each of the following.

(a) State the definition of the limit of a sequence: $\lim_{n \rightarrow \infty} a_n = L$. (41)

(b) State the definition of the sum of a series: $\sum_{n=1}^{\infty} a_n = S$. (41)

(c) Find the limit of the sequence $\{a_n\} = \{\tan^{-1} n\}$. Explain your answer. (35, 41)

10. Do each of the following.

(a) For what values of x does $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{5^n(n+5)}$ converge? (46)

(b) Compute a MacLaurin series for $\sin(-x^3)$ and $\int \sin(-t^3) dt$. (34, 47)