CALCULUS COMPREHENSIVE EXAM

Summer 2004, Prepared by Dr. Robert Gardner

August 13, 2004

NAME ______________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. **(a,b)** Suppose the interval $[a, b]$ is a subset of the domain of the function f. What is the definition of "f is continuous at point c where $c \in (a, b)$"? What is the definition of "f is continuous on the interval $[a, b]$"? **(4)**

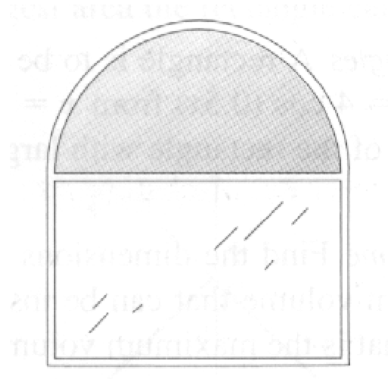
(c) Use the definition from **(a)** and **(b)** to show that $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$. **(4)**

2. **(a)** What does it mean for $f(x)$ to be implicit to the equation $F(x, y) = 0$?

(b) Find y' if $\sin(xy) = \ln(x \cot y)$. **(8, 10, 31, 34)**

3. State the Extreme Value Theorem for Continuous Functions. Consider $f(x) = x^{2/3}$ on the interval $[-2, 3]$. Find the absolute extrema of f on this interval.

4. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed (call it p). Find the proportions of the window that will admit the most light. **(18)**



5. **(a)** State the Fundamental Theorem of Calculus (both parts). **(23)**

(b) Evaluate $\int_1^e \ln x\,dx$ (HINT: use parts) and indicate with a star ($*$) where you are applying the Fundamental Theorem. **(23, 24, 31)**

6. **(a)** State the definition of $\ln x$ (in terms of definite integrals). **(29)**

(b) Use the definition to prove that $\ln ab = \ln a + \ln b$ ($a > 0$, $b > 0$). **(23, 29)**

7. Do the following:

(a) Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$. (Show all work!) **(30, 37)**

(b) Evaluate $\int_0^2 \frac{dx}{(x-1)^2}$. **(39)**

(c) Use the Integral Test to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

8. **(a)** Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\lim_{n\to\infty} a_n = L$." **(41)**

(b) Let $\sum_{n=1}^{\infty} a_n$ be a series. Define *partial sum* of the series and define "$\left(\sum_{n=1}^{\infty} a_n\right) = L$." **(41)**

9. **(a)** For a given x value, the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? **(46)**

(b) What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{(3x+6)^n}{n!}$. **(46)**

10. Compute a Taylor series for e^{-x^2} and $\int_0^x e^{-t^2}\,dt$. **(47)**