CALCULUS COMPREHENSIVE EXAM

Spring 2013b Prepared by Dr. Robert Gardner

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NAME _____ Start time: _____ End time: _____

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Do each of the following:

(a) State the definition of the limit of a function (i.e., what does $\lim_{x\to a} f(x) = L$ mean?). **(1)**

(b) Use the definition from (a) to prove that if $\lim_{x\to a} f(x) = L$ and $c \in \mathbb{R}$, then $\lim_{x\to a} cf(x) = cL$. **(2)**

2. Do each of the following:

(a) State the definition of *derivative* of a function f. **(6)**

(b) Use the definition to differentiate $f(x) = \dfrac{1}{\sqrt{x}}$. **(2, 6, 8)**

3. Find the volume of the largest right circular cylinder which can be inscribed in a sphere of radius R. The volume of a right circular cylinder of radius r and height h is $V = \pi r^2 h$ and the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. **(12, 16, 18)**.

4. Do each of the following **(23, 24)**

(a) State the two parts of the Fundamental Theorem of Calculus. **(23)**

(b) Use the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^e \ln x \, dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem. Hint: Integrate by parts. **(23, 24, 31)**

5. Do each of the following.

(a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\displaystyle\int_a^b f(x)\, dx$. **(21)**

(b) Suppose the function $y = f(x)$ is implicit to the equation $x^2 + y^2 = 1$. Find y'' at $(\sqrt{2}/2, \sqrt{2}/2)$. **(10)**

6. Do each of the following:

(a) Use the definition of $y = \tan^{-1} x$ (in terms of the tangent function) and implicit differentiation to find $y' = \dfrac{d}{dx}[\tan^{-1} x]$. **(10, 28, 35)**

(b) Find the length of the curve given by the equation $y = \displaystyle\int_0^x \sqrt{\sec^4 t - 1}\, dt$ for $-\pi/4 \le x \le \pi/4$. **(23, 27)**

7. Do each of the following. Respect the calculus!

(a) Evaluate $\displaystyle\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$. **(31, 37)**

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$. **(39)**

8. Do each of the following:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? **(46)**

(b) What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$? (**45**)

9. Do each of the following.

(a) State the definition of the limit of a sequence: $\displaystyle\lim_{n \to \infty} a_n = L$. **(41)**

(b) State the definition of the sum of a series: $\displaystyle\sum_{n=1}^{\infty} a_n = S$. **(41)**

(c) Find the limit of the sequence $\{a_n\} = \{\tan^{-1} n\}$. Explain your answer. (**35, 41**)

10. Do each of the following.

(a) For what values of x does $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{5^n (n+5)}$ converge? **(46)**

(b) Compute a MacLaurin series for $\sin(-x^3)$ and $\displaystyle\int \sin(-t^3)\, dt$. **(34, 47)**