CALCULUS COMPREHENSIVE EXAM

Spring 2017b, Prepared by Dr. Robert Gardner

April 21, 2017

NAME ____________________________ Start Time _______ End Time: _______

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.**

To address potential academic misconduct during the test, I will wander the room and may request to see the progress of your work on the test while you are taking it. You are not allowed to access your phone during the test. You are not allowed to stop during a test to go to the bathroom, unless you have presented a documented medical need beforehand.

You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____.

1. Do each of the following **(1)**:

(a) State the definition of the limit of a function (i.e., what does $\lim_{x\to a} f(x) = L$ mean?).

(b) Use the definition of limit to prove that $\lim_{x\to a}(-5x+3) = -5a+3$.

2. Do each of the following **(3, 33)**:

(a) State the Sandwich Theorem (sometimes called the "Squeeze Theorem") for the limit of a function.

(b) Use the fact that $\sin\theta < \theta < \tan\theta$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ (WARNING: This is a two sided limit and the inequality is only known to hold for $\theta \in (0, \pi/2)$.)

3. Do each of the following **(5)**:

(a) State the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to *prove* that $f(x) = x^5 - 5x - 1$ has a real root (be sure to include all necessary hypotheses).

4. Do each of the following **(23, 24)**:

(a) State the two parts of the Fundamental Theorem of Calculus.

(b) Use the Fundamental Theorem of Calculus to evaluate $\int_0^1 xe^{x^2}\,dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

5. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x)\,dx$. **(21)**

(b) Explain the difference between a definite integral and an indefinite integral (if any). **(20, 23)**

6. Do each of the following **(32, 37)**:

(a) If f is continuous on $[a, c) \cup (c, b]$ then state the definition of $\int_a^b f(x)\,dx$. That is, how do you integrate over a discontinuity? You may assume the usual definition for integrals of continuous functions has been established.

(b) Evaluate $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$.

7. Do each of the following:

(a) Use the definition of $y = \tan^{-1} x$ (in terms of the tangent function) and implicit differentiation to find $y' = \dfrac{d}{dx}[\tan^{-1} x]$. **(10, 28, 35)**

(b) Find the length of the curve given by the equation $y = \displaystyle\int_0^x \sqrt{\sec^4 t - 1}\,dt$ for $-\pi/4 \leq x \leq \pi/4$. **(23, 27)**

8. Do each of the following **(39, 41, 43)**:

(a) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2}\,dx$.

(b) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\lim_{n\to\infty} a_n = L$."

(c) Use the Integral Test to show that the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. Do each of the following **(46)**:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e., on what types of sets might the series converge conditionally, converge absolutely, or diverge)?

(b) What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$?

10. Find a MacLaurin Series for $f(x) = e^{2x} - e^x$ (show your work). Show and/or explain why the series converges absolutely for all x. Use the series to calculate $\displaystyle\lim_{x\to 0} \frac{e^{2x} - e^x}{x}$. **(31, 45, 46, 47)**