CALCULUS COMPREHENSIVE EXAM

Srping 2005, Prepared by Dr. Robert Gardner

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NAME________________________ STUDENT NUMBER____________________________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 though 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______. There is a three hour time limit.

1. Do each of the following **(3, 33)**:

(a) State the Sandwich Theorem (sometimes called the "Squeeze Theorem") for the limit of a function.

(b) Use the facts that $\sin\theta < \theta < \tan\theta$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (WARNING: This is a two sided limit and the inequality is only known to hold for $\theta \in (0, \pi/2)$.)

2. Do each of the following **(5)**:

(a) State the Intermediate Value Theorem.

(b) Prove that $\cos x = x$ for some x.

3. Do each of the following **(8, 10, 31, 35)**:

(a) State the Chain Rule (with all hypotheses).

(b) What does it mean for $f(x)$ to be implicit to the equation $F(x, y) = 0$?

(c) Find $\frac{dy}{dx}$: $\tan^{-1}(\ln y) = e^{x^2}$.

4. A silo is to be made in the form of a cylinder surmounted by a hemisphere. The cost of construction per square foot of surface area is twice as great for the hemisphere as for the cylinder. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be minimum. Neglect the thickness of the silo and waste in construction. (Ignore the top and the bottom of the cylinder.) HINT: The surface area of a sphere is $A = 4\pi r^2$, the volume of a sphere is $V = \frac{4}{3}\pi r^3$, and the volume of a cylinder is $V = \pi r^2 h$.

5. Do each of the following **(21 23)**:

(a) State the definition of *partition*, *norm* of a partition, *Reimann sum*, and *definite integral* for $\int_a^b f(x)\,dx$.

(b) State the two parts of the Fundamental Theorem of Calculus.

6. Do each of the following **(29)**:

(a) State the definition of $\ln x$ (using integrals).

(b) Use the definition from part **(a)** to *prove* that $\ln(ab) = \ln(a) + \ln(b)$.

7. Do each of the following **(24, 37, 38, 39)**:

(a) Evaluate $\int_{-1}^{1} \frac{1}{x^2}\, dx$. BE CAREFUL and don't write anything wrong!

(b) Evaluate $\int_{1}^{3} \frac{1}{1-x^2}\, dx$.

(c) Evaluate $\lim_{x\to 0} \left(\csc x - \frac{1}{x} \right)$.

8. Do each of the following **(41, 43)**:

(a) State the definition of the limit of a sequence: $\lim_{n\to\infty} a_n = L$.

(b) State the definition of the sum of a series: $\sum_{n=1}^{\infty} a_n = S$.

(c) Use the Integral Test to show that a p-series with $p > 1$ converges.

9. Do each of the following **(45, 46)**:

(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge).

(b) For what values of x does $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$ converge?

10. Compute a Taylor series for $f(x) = \ln x$ about $a = 1$. Find the radius of convergence. **(46, 47)**