CALCULUS COMPREHENSIVE EXAM

Spring 2003, Prepared by Dr. Robert Gardner

April 11, 2003

NAME ______________________________ STUDENT NUMBER ________________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. Do each of the following **(4)**:

   a. What does it mean for function f to be continuous at point a?

   b. What does it mean for f to be continuous on interval $[a, b]$?

   c. Show that $f(x) = \sqrt{1 - x^2}$ is (or is not) continuous on $[-1, 1]$.

2. State the Intermediate Value Theorem. Prove that $\cos x = x$ for some x. **(5)**

3. Consider $f(x) = \dfrac{x^3}{x(1 - x^2)}$. Find the first and second derivative of f, find where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$. (**8, 14, 15, 16, 17**)

4. Find the volume of the largest right circular cylinder which can be inscribed in a sphere of radius R. The volume of a right circular cylinder of radius r and height h is $V = \pi r^2 h$ and the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. **(12, 16, 18)**.

5. Do each of the following **(23, 24)**:

   a. State the two parts of the Fundamental Theorem of Calculus.

   b. Use the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^1 x \sin x \, dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

6. **a.** State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\displaystyle\int_a^b f(x) \, dx$.

   b. Explain the difference between a definite integral and an indefinite integral (if any). (**20, 21, 23**)

7. Find the length of the curve given by the equation $y = \displaystyle\int_0^x \sqrt{\sec^4 t - 1} \, dt$ for $-\pi/4 \leq x \leq \pi/4$. **(23, 27)**

8. (a) Use the definition of $y = \sin^{-1} x$ (in terms of the sine function) and implicit differentiation to find $y' = \dfrac{d}{dx}\left[\sin^{-1} x\right]$.

(b) Evaluate $\displaystyle\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$. (**28, 34, 35**)

9. **a.** Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\lim_{n\to\infty} (a_n) = L$."

b. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series. Define *partial sum* of the series and define "$\left(\displaystyle\sum_{n=1}^{\infty} a_n\right) = L$." (**41**)

10. State L'Hôpital's Rule for an ∞/∞ indeterminate form. Use L'Hôpital's Rule to show $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$. **(31, 37)**

11. Do each of the following **(38, 39, 41)**:

a. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$.

b. Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$.

12. **a.** For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)?

b. What is the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$? (**46**)