CALCULUS COMPREHENSIVE EXAM

Spring 2004, Prepared by Dr. Robert Gardner

January 30, 2004

NAME ______________________________ STUDENT NUMBER ______________________

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. Do each of the following:

(a) State the definition of the limit of a function (that is, what does $\lim_{x \to a} f(x) = L$ mean?).

(b) Use the definition of to prove that $\lim_{x \to a} mx + b = ma + b$. **(1)**

2. State the Intermediate Value Theorem. Prove that $\cos x = x$ for some x. **(5)**

3. Consider $f(x) = \dfrac{x^3}{x(1-x^2)}$. Find the first and second derivative of f, find where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$. **(8, 14, 15, 16, 17)**

4. The intensity of illumination at any point is proportional to the product of the strength of the light source and the inverse of the square of the distance from the source. If two sources of relative strengths a and b are a distance c apart, at what point on the line joining them will the intensity be minimum? Assume the intensity at any point is the sum of intensities from the two sources.

5. Do each of the following:

(a) State the Fundamental Theorem of Calculus (both parts). **(23)**

(b) Use the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^1 \frac{1+x}{1+x^2}\,dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem. **(24, 35)**

6. Do each of the following **(29)**:

(a) State the definition of $\ln x$ (using integrals).

(b) Use the definition to *prove* that $\ln x^n = n \ln x$ for n rational.

7. Do each of the following **(33, 34, 35)**:

(a) Evaluate $\lim_{y \to 0} \frac{\sin 3y}{4y}$.

(b) Evaluate $\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$.

(c) Evaluate $\int \frac{\sec^2 x \, dx}{\sqrt{1 - \tan^2 x}}$.

8. Do each of the following **(39, 41, 43)**:

(a) If f is continuous on $[a, c) \bigcup (c, b]$ then state the definition of $\int_a^b f(x) \, dx$. That is, how do you integrate over a discontinuity? You may assume the usual definition for integrals of continuous functions has been established.

(b) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\lim_{n \to \infty} a_n = L$."

(c) Use the Integral Test to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

9. Do each of the following **(46)**:

(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge?).

(b) What is the radius of convergence for $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2 + 3}}$ (give detailed reasons for your answer).

10. Do each of the following **(44, 47)**:

(a) Use the MacLaurin series for e^x to find a series for $\int e^{-x^2} \, dx$.

(b) Estimate $\int_0^1 e^{-x^2} \, dx$ to the nearest 0.001 and explain why you know your answer has this level of accuracy.