

# CALCULUS COMPREHENSIVE EXAM

Spring 2003, Prepared by Dr. Robert Gardner

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NAME \_\_\_\_\_ STUDENT NUMBER \_\_\_\_\_

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: \_\_\_\_\_ and \_\_\_\_\_.

1. Do each of the following **(4)**:

- What does it mean for function  $f$  to be continuous at point  $a$ ?
- What does it mean for  $f$  to be continuous on interval  $[a, b]$ ?
- Show that  $f(x) = \sqrt{1 - x^2}$  is (or is not) continuous on  $[-1, 1]$ .

2. State the Intermediate Value Theorem. Prove that  $\cos x = x$  for some  $x$ . **(5)**

3. Consider  $f(x) = \frac{x^3}{x(1 - x^2)}$ . Find the first and second derivative of  $f$ , find where  $f$  is increasing/decreasing, find where  $f$  is concave up/concave down, find the asymptotes of the graph of  $f$ , find the extrema of  $f$ , and graph  $y = f(x)$ . **(8, 14, 15, 16, 17)**

4. Find the volume of the largest right circular cylinder which can be inscribed in a sphere of radius  $R$ . The volume of a right circular cylinder of radius  $r$  and height  $h$  is  $V = \pi r^2 h$  and the volume of a sphere of radius  $R$  is  $V = \frac{4}{3}\pi R^3$ . **(12, 16, 18)**.

5. Do each of the following **(23, 24)**:

a. State the two parts of the Fundamental Theorem of Calculus.

b. Use the Fundamental Theorem of Calculus to evaluate  $\int_0^1 x \sin x \, dx$  and indicate with a star (\*) where you are applying the Fundamental Theorem.

6. a. State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for  $\int_a^b f(x) \, dx$ .

b. Explain the difference between a definite integral and an indefinite integral (if any). **(20, 21, 23)**

7. Find the length of the curve given by the equation  $y = \int_0^x \sqrt{\sec^4 t - 1} \, dt$  for  $-\pi/4 \leq x \leq \pi/4$ . **(23, 27)**

8. (a) Use the definition of  $y = \sin^{-1} x$  (in terms of the sine function) and implicit differentiation to find  $y' = \frac{d}{dx} [\sin^{-1} x]$ .
- (b) Evaluate  $\int_0^{3\sqrt{2}/4} \frac{dx}{\sqrt{9-4x^2}}$ . (**28, 34, 35**)
9. **a.** Let  $\{a_n\} = \{a_1, a_2, a_3, \dots\}$  be a sequence of real numbers. Define “ $\lim_{n \rightarrow \infty} (a_n) = L$ .”
- b.** Let  $\sum_{n=1}^{\infty} a_n$  be a series. Define *partial sum* of the series and define “ $\left(\sum_{n=1}^{\infty} a_n\right) = L$ .” (**41**)
10. State L'Hôpital's Rule for an  $\infty/\infty$  indeterminate form. Use L'Hôpital's Rule to show  $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$ . (**31, 37**)
11. Do each of the following (**38, 39, 41**):
- a.** Evaluate  $\int_{-\infty}^{\infty} \frac{2x dx}{(x^2 + 1)^2}$ .
- b.** Evaluate  $\int_{-1}^1 \frac{1}{x^2} dx$ .
12. **a.** For a given  $x$  value, the power series  $\sum_{n=0}^{\infty} c_n(x-a)^n$  may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)?
- b.** What is the radius of convergence of  $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ ? (**46**)