CALCULUS COMPREHENSIVE EXAM
Spring 2002, Prepared by Dr. Robert Gardner
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NAME ____________________ STUDENT NUMBER ____________________

Be clear and give all details. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics from the Study Guide covered in that problem. You may omit one problem from numbers 1 through 6 (which contains Calculus 1 material) and one problem from numbers 7 through 12 (which contains Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. (a) State the definition of the limit of a function (that is, what does \( \lim_{x \to a} f(x) = L \) mean?). (b) Use the definition to prove that \( \lim_{x \to a} mx + b = ma + b \). (1)

2. State the definition of “f is continuous on \([a, b]\)” and prove that \( f(x) = \sqrt{1 - x^2} \) is continuous on \([-1, 1]\). (2, 4)

3. Consider \( f(x) = \frac{x^3}{x(1 - x^2)} \). Find the first and second derivative of \( f \), find where \( f \) is increasing/decreasing, find where \( f \) is concave up/concave down, find the asymptotes of the graph of \( f \), find the extrema of \( f \), and graph \( y = f(x) \). (8, 14, 15, 16, 17)

4. Sand falls from a conveyor belt at the rate of 10 m\(^3\)/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast is the height changing when the pile is 4 m high? HINT: The volume \( V \) of a (right circular) cone of base radius \( r \) and height \( h \) is \( V = \frac{1}{3}\pi r^2 h \). (23, 24)

5. State both parts of the Fundamental Theorem of Calculus. Evaluate \( \int_1^2 \frac{(\ln x)^7}{x} \, dx \) and indicate with a star (*) where you have used the Fundamental Theorem of Calculus in your computation.

6. (a) State the definition of partition, norm of a partition, Riemann sum, and definite integral for \( \int_a^b f(x) \, dx \).
   (b) Explain the difference between a definite integral and an indefinite integral (if any). (20, 21, 23)

7. State the definition of \( \ln x \). Use the definition to prove that \( \ln ab = \ln a + \ln b \). (23, 29)

8. Find the length of \( y = x^2 \) for \( x \in [0, 1] \). (27, 34)
9. (a) Use the definition of \( y = \sin^{-1} x \) (in terms of the sine function) and implicit differentiation to find \( y' = \frac{d}{dx} [\sin^{-1} x] \).

(b) Evaluate \( \int_{0}^{3\sqrt{2}/4} \frac{dx}{\sqrt{9 - 4x^2}} \).

10. (a) State the definition of the limit of a sequence: \( \lim_{n \to \infty} a_n = L \).

(b) State the definition of the sum of a series: \( \sum_{n=1}^{\infty} a_n = S \).

(c) Find the limit of the sequence \( \{a_n\} = \{\tan^{-1} n\} \). (35, 41)

11. Consider \( \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right) \frac{x^n}{n} \). Find the interval of convergence, radius of convergence, and values of \( x \) for which the convergence is absolute or conditional. (44, 45, 46)

12. (a) Find a power series for \( f(x) = e^{-x^2} \).

(b) Use your power series to calculate \( \int_{0}^{1} e^{-x^2} \, dx \) (as a series). How many nonzero terms of the series representation of this integral must be summed to approximate the integral to the nearest \( \frac{1}{100} \)? (44, 47)