CALCULUS COMPREHENSIVE EXAM
Summer 2014, Prepared by Dr. Robert Gardner
August 1, 2014

NAME ___________________________ Start Time _______ End Time: _______

Be clear and give all details. Use all symbols correctly (such as equal signs). The numbers in bold
faced parentheses indicate the number of the topics covered in that problem from the Study Guide.
No calculators and turn off your cell phones! You may omit one problem from numbers 1
through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which
contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. (a) State the definition of the limit of a function (i.e., what does \( \lim_{x \to a} f(x) = L \) mean?).
   
   (b) Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then \( \lim_{x \to a} (f(x) - g(x)) = L - M \) (1,2)

2. (a) State L'Hôpital's rule.
   
   (b) Determine \( \lim_{x \to 0^+} (1 - 2x)^{3/x} \) (37)

3. Do each of the following (5, 13):
   
   (a) State the Intermediate Value Theorem.
   
   (b) State the Mean Value Theorem.
   
   (c) Prove that \( f(x) = \sin(x) + 2x - 1 \) has exactly one real root.

4. (a) State the Fundamental Theorem of Calculus (both parts). (23)
   
   (b) Evaluate \( \int_0^1 \tan^{-1}(x) \, dx \) (HINT: Use parts) and indicate with a star (*) where you have
   used the Fundamental Theorem of Calculus in your computations. (24, 31)

5. (a) State the definition of partition, norm of a partition, Riemann sum, and definite integral for
   \( \int_a^b f(x) \, dx \). (21)
   
   (b) Explain the difference between a definite integral and an indefinite integral (if any). (20, 23)
6. Do each of the following:

(a) Use the definition of \( y = \tan^{-1} x \) (in terms of the tangent function) and implicit differentiation to find \( y' = \frac{d}{dx}[\tan^{-1} x] \). (10, 28, 35)

(b) Find the length of the curve given by the equation \( y = \int_{0}^{x} \sqrt{\sec^4 t - 1} \, dt \) for \(-\pi/4 \leq x \leq \pi/4\). (23, 27)

7. Do each of the following. Respect the calculus!

(a) Evaluate \( \lim_{x \to 0^+} \left( 1 + \frac{1}{x} \right)^x \). (31, 37)

(b) Evaluate \( \int_{-1}^{1} \frac{1}{x^2} \, dx \). (39)

8. Do each of the following.

(a) State the definition of the limit of a sequence: \( \lim_{n \to \infty} a_n = L \). (41)

(b) State the definition of the sum of a series: \( \sum_{n=1}^{\infty} a_n = S \). (41)

(c) Determine whether the series \( \sum_{n=1}^{\infty} n e^{-2n} \) is convergent or divergent. You may use any test, but you must check the hypothesis of any test you use. (45)

9. Do each of the following:

(a) For a given \( x \) value, the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? (46)

(b) For what values of \( x \) does \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x - 5)^n}{5^n(n + 5)} \) converge? (46)

10. Do each of the following:

(a) Use the Maclaurin Series for \( e^x \) to find a series for \( \int e^{-x^2} \, dx \). (30, 46)

(b) Estimate \( \int_{0}^{1} e^{-x^2} \, dx \) to the nearest 0.001 and explain why you know your answer has this level of accuracy. (44, 47)