CALCULUS COMPREHENSIVE EXAM

Summer 2014, Prepared by Dr. Robert Gardner

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NAME ________________________________ Start Time ________ End Time: ________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. (a) State the definition of the limit of a function (i.e., what does $\lim_{x \to a} f(x) = L$ mean?).

(b) Prove that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a}(f(x) - g(x)) = L - M$ **(1,2)**

2. (a) State L'Hôpital's rule.

(b) Determine $\lim_{x \to 0^+} (1 - 2x)^{3/x}$ **(37)**

3. Do each of the following **(5, 13)**:

(a) State the Intermediate Value Theorem.

(b) State the Mean Value Theorem.

(c) Prove that $f(x) = \sin(x) + 2x - 1$ has exactly one real root.

4. (a) State the Fundamental Theorem of Calculus (both parts). **(23)**

(b) Evaluate $\int_0^1 \tan^{-1}(x)\, dx$ (HINT: Use parts) and indicate with a star (*) where you have used the Fundamental Theorem of Calculus in your computations. **(24, 31)**

5. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x)\, dx$. **(21)**

(b) Explain the difference between a definite integral and an indefinite integral (if any). **(20, 23)**

6. Do each of the following:

(a) Use the definition of $y = \tan^{-1} x$ (in terms of the tangent function) and implicit differentiation to find $y' = \dfrac{d}{dx}[\tan^{-1} x]$. **(10, 28, 35)**

(b) Find the length of the curve given by the equation $y = \int_0^x \sqrt{\sec^4 t - 1}\, dt$ for $-\pi/4 \le x \le \pi/4$. **(23, 27)**

7. Do each of the following. Respect the calculus!

(a) Evaluate $\displaystyle\lim_{x\to 0^+} \left(1 + \frac{1}{x}\right)^x$. **(31, 37)**

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$. **(39)**

8. Do each of the following.

(a) State the definition of the limit of a sequence: $\displaystyle\lim_{n\to\infty} a_n = L$. **(41)**

(b) State the definition of the sum of a series: $\displaystyle\sum_{n=1}^{\infty} a_n = S$. **(41)**

(c) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ is convergent or divergent. You may use any test, but you must check the hypothesis of any test you use. **(45)**

9. Do each of the following:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? **(46)**

(b) For what values of x does $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{5^n(n+5)}$ converge? **(46)**

10. Do each of the following:

(a) Use the MacLaurin Series for e^x to find a series for $\displaystyle\int e^{-x^2} dx$. **(30, 46)**

(b) Estimate $\displaystyle\int_0^1 e^{-x^2} dx$ to the nearest 0.001 and explain why you know your answer has this level of accuracy. **(44, 47)**