CALCULUS COMPREHENSIVE EXAM

Spring 2018b, Prepared by Dr. Robert Gardner

April 20, 2018

NAME ________________________________ Start Time _________ End Time: _________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. **(a)** State the definition of the limit of a function (i.e., what does $\lim_{x\to a} f(x) = L$ mean?).

(b) Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) - g(x)) = L - M$ **(1,2)**

2. Do each of the following **(3)**:

(a) State the Sandwich Theorem (also called the Squeeze Theorem) for the limit of a function.

(b) Use the fact that $\sin(\theta) < \theta < \tan(\theta)$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$. WARNING: This is a two sided limit and the inequality is known to hold only for $\theta \in (0, \pi/2)$.

3. Do each of the following **(5, 13)**:

(a) State the Intermediate Value Theorem.

(b) State the Mean Value Theorem.

(c) Prove that $f(x) = \sin(x) + 2x - 1$ has exactly one real zero.

4. (a) State the Fundamental Theorem of Calculus (both parts). **(23)**

(b) Evaluate $\int_1^e \ln x\, dx$ (HINT: Use parts) and indicate with a star (*) where you have used the Fundamental Theorem of Calculus in your computations. **(24, 31)**

5. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x)\, dx$. **(21)**

(b) Explain the difference between a definite integral and an indefinite integral (if any). **(20, 23)**

6. **(a)** Use the definition of $y = \sin^{-1} x$ (in terms of the sine function) and implicit differentiation to find $y' = \dfrac{d}{dx}[\sin^{-1} x]$. **(35)**

(b) Evaluate $\displaystyle\int \frac{dx}{\sqrt{5 - 4x - x^2}}$. **(28, 34, 35)**

7. Do each of the following. Respect the calculus!

(a) Evaluate $\displaystyle\lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x$. **(31, 37)**

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$. **(39)**

8. Do each of the following.

(a) State the definition of the limit of a sequence: $\displaystyle\lim_{n \to \infty} a_n = L$. **(41)**

(b) State the definition of the sum of a series: $\displaystyle\sum_{n=1}^{\infty} a_n = S$. **(41)**

(c) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ is convergent or divergent. You may use any test, but you must check the hypothesis of any test you use. **(45)**

9. Do each of the following:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? **(46)**

(b) Show that for $0 < p < 1$, the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. **(38, 43)**

10. Compute a MacLaurin series for e^{-x^2} and $\displaystyle\int_0^x e^{-t^2}\, dt$. **(47)**