CALCULUS COMPREHENSIVE EXAM
Spring 2018b, Prepared by Dr. Robert Gardner
April 20, 2018

NAME ____________________________ Start Time _______ End Time: _______

Be clear and give all details. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. No calculators and turn off your cell phones! Use the paper provided and only write on one side. You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: _______ and _______.

1. (a) State the definition of the limit of a function (i.e., what does \( \lim_{x \to a} f(x) = L \) mean?).
   (b) Prove that if \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \), then \( \lim_{x \to a} (f(x) - g(x)) = L - M \) (1,2)

2. Do each of the following (3):
   (a) State the Sandwich Theorem (also called the Squeeze Theorem) for the limit of a function.
   (b) Use the fact that \( \sin(\theta) < \theta < \tan(\theta) \) for \( \theta \in (0, \pi/2) \) to show that \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \).
   WARNING: This is a two sided limit and the inequality is known to hold only for \( \theta \in (0, \pi/2) \).

3. Do each of the following (5, 13):
   (a) State the Intermediate Value Theorem.
   (b) State the Mean Value Theorem.
   (c) Prove that \( f(x) = \sin(x) + 2x - 1 \) has exactly one real zero.

4. (a) State the Fundamental Theorem of Calculus (both parts). (23)
   (b) Evaluate \( \int_1^e \ln x \, dx \) (HINT: Use parts) and indicate with a star (*) where you have used the Fundamental Theorem of Calculus in your computations. (24, 31)

5. (a) State the definition of partition, norm of a partition, Riemann sum, and definite integral for \( \int_a^b f(x) \, dx \). (21)
   (b) Explain the difference between a definite integral and an indefinite integral (if any). (20, 23)
6. (a) Use the definition of \( y = \sin^{-1} x \) (in terms of the sine function) and implicit differentiation to find \( y' = \frac{d}{dx}[\sin^{-1} x] \). (35) 

(b) Evaluate \( \int \frac{dx}{\sqrt{5 - 4x - x^2}} \). (28, 34, 35)

7. Do each of the following. Respect the calculus!

(a) Evaluate \( \lim_{x \to 0^+} \left(1 + \frac{1}{x}\right)^x \). (31, 37)

(b) Evaluate \( \int_{-1}^{1} \frac{1}{x^2} \, dx \). (39)

8. Do each of the following.

(a) State the definition of the limit of a sequence: \( \lim_{n \to \infty} a_n = L \). (41)

(b) State the definition of the sum of a series: \( \sum_{n=1}^{\infty} a_n = S \). (41)

(c) Determine whether the series \( \sum_{n=1}^{\infty} ne^{-2n} \) is convergent or divergent. You may use any test, but you must check the hypothesis of any test you use. (45)

9. Do each of the following:

(a) For a given \( x \) value, the power series \( \sum_{n=0}^{\infty} c_n(x-a)^n \) may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)? (46)

(b) Show that for \( 0 < p < 1 \), the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) diverges. (38, 43)

10. Compute a MacLaurin series for \( e^{-x^2} \) and \( \int_{0}^{x} e^{-t^2} \, dt \). (47)