CALCULUS COMPREHENSIVE EXAM

Spring 2015, Prepared by Dr. Robert Gardner

February 13, 2015

NAME ________________________________ Start Time _________ End Time: _________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators and turn off your cell phones!** Use the paper provided and **only write on one side.** You may omit one problem from numbers 1 through 5 (which contain Calculus 1 material) and one problem from numbers 6 through 10 (which contain Calculus 2 material). Indicate which two problems you are omitting: ______ and ______.

1. Do each of the following **(1)**:

(a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).

(b) Prove that $\lim_{x\to a} mx + b = ma + b$ for $m \neq 0$.

2. Do each of the following **(3, 33)**:

(a) State the Sandwich Theorem (sometimes called the "Squeeze Theorem") for the limit of a function.

(b) Use the fact that $\sin\theta < \theta < \tan\theta$ for $\theta \in (0, \pi/2)$ to show that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (WARNING: This is a two sided limit and the inequality is only known to hold for $\theta \in (0, \pi/2)$.)

3. Do each of the following **(5)**:

(a) State the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to *prove* that $f(x) = x^5 - 5x - 1$ has a real root (be sure to include all necessary hypotheses).

4. Do each of the following **(17)**:

(a) Give the definition of "f has a vertical asymptote at $x = a$."

(b) Give the definition of "f has a horizontal asymptote of $y = k$."

(c) Find the asymptotes of $y = \frac{x+5}{8-2x}$ and graph.

5. Do each of the following**(23, 24, 35)**:

(a) State the two parts of the Fundamental Theorem of Calculus.

(b) Use the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^{\pi/2} x\sin x\,dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

6. Do each of the following **(32, 37)**:

(a) If f is continuous on $[a,c)\cup(c,b]$ then state the definition of $\int_a^b f(x)\,dx$. That is, how do you integrate over a discontinuity? You may assume the usual definition for integrals of continuous functions has been established.

(b) Evaluate $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$.

7. Evaluate **(38, 39)**:

(a) $\displaystyle\lim_{x\to 0^+} x^x$.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ (use all notation correctly and don't write things that don't make sense).

8. Do each of the following **(41)**:

(a) Let $\{a_n\} = \{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Define "$\displaystyle\lim_{n\to\infty} a_n = L$."

(b) Let $\{a_n\}$ and $\{b_n\}$ be sequences with $\displaystyle\lim_{n\to\infty} a_n = L$ and $\displaystyle\lim_{n\to\infty} b_n = M$. Use the definition of from part (a) to prove $\displaystyle\lim_{n\to\infty}(a_n + b_n) = L + M$."

9. Do each of the following **(46)**:

(a) For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge?).

(b) What is the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2+3}}$ (give detailed reasons for your answer).

10. Find a MacLaurin Series for $f(x) = e^{2x} - e^x$ (show your work). Show and/or explain why the series converges absolutely for all x. Use the series to calculate $\displaystyle\lim_{x\to 0} \frac{e^{2x}-e^x}{x}$. **(31, 45, 46, 47)**