CALCULUS COMPREHENSIVE EXAM
SPRING 1997, Prepared by Dr. Robert Gardner

NAME _____ STUDENT NUMBER _____

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide.

You may omit one problem from numbers 1 through 7 (which contain material from Calculus 1) and one problem from numbers 8 through 12 (which contain material from Calculus 2). Indicate which two problems you are omitting: _____ and _____.

1. Do each of the following **(1)**:

a. Give the definition of "the limit as x approaches a of $f(x)$ is L: $\lim_{x\to a} f(x) = L$."

b. Use the definition to *prove* that for $m \neq 0$, $\lim_{x\to a}(mx + b) = ma + b$.

2. Do each of the following (**5**):

a. State the Intermediate Value Theorem.

b. Use the Intermediate Value Theorem to *prove* that $f(x) = x^3 - 3x - 1$ has a real root (be sure to include all necessary hypotheses).

3. Do each of the following **(8, 31, 34)**:

a. Clearly state the Product Rule, Quotient Rule, and the Chain Rule.

b. Differentiate (you need not simplify your answer): $f(x) = \ln\left(\dfrac{\sin(x^2) - 5x}{e^x + \sec x}\right)$.

c. Differentiate (you need not simplify your answer): $f(x) = \sqrt{\csc(\cot(e^{2x}))}$.

4. Do each of the following **(14)**:

a. Clearly state the relationship between the increasing-ness/decreasing-ness of a function and its first derivative.

b. Find the intervals on which $f(x) = x^{1/3}(x - 4)$ is increasing/decreasing and graph f.

5. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ and the volume of a sphere is $V = \frac{4}{3}\pi r^3$. (**18**)

6. Do each of the following **(23, 24)**:

a. State the two parts of the Fundamental Theorem of Calculus.

b. Use the Fundamental Theorem of Calculus to evaluate $\displaystyle\int_0^1 xe^x\,dx$ and indicate with a star $(*)$ where you are applying the Fundamental Theorem.

7. The region in the first quadrant bounded by $y = x^2$, the y-axis, and the line $y = 1$ is revolved about the line $x = 2$. Find the resulting volume. (**26**)

8. Do each of the following **(29)**:

a. State the definition of $\ln x$ (using integrals).

b. Use the definition to *prove* that: $\ln x^n = n \ln x$ for n rational.

9. Do each of the following **(24, 35, 44)**:

a. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$.

b. State the definition of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ (include the range of each function).

c. State the definition of an Alternating Series and approximate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ to within 0.25 and explain why your approximation is this accurate.

10. Evaluate **(37, 38, 39)**:

a. $\displaystyle\lim_{x \to 0^+} x^x$.

b. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx$ (use all notation correctly and don't write things that don't make sense).

c. $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$.

11. Do each of the following **(46)**:

a. For a given x value, the power series $\displaystyle\sum_{n=0}^{\infty} c_n (x - a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e. on what types of sets might the series converge conditionally, converge absolutely, or diverge)?

b. What is the radius of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\sqrt{n^2 + 3}}$ (give detailed reasons for your answer).

12. Do each of the following **(42, 43)**:

a. State the Test for Divergence.

b. What is a p-series and for what values of p does a p-series converge/diverge?

c. Use the Integral Test to show that the harmonic series diverges.