LINEAR ALGEBRA COMPREHENSIVE EXAM

<Semester,Yr> Prepared by <Faculty Member>

Exam Date

NAME Exam Key ______________________ STUDENT NUMBER ______________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} \in \mathbb{R}^3$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and express the solution as the translation of a vector space. (**A3, A4, A5, D6**)

Solution: The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{array}\right]$$

row reduces as follows:

$$\xrightarrow[-7R_1+R_3]{-4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -2 \\ 0 & -6 & -12 & -4 \end{array}\right] \xrightarrow[-2R_2+R_3]{2/3R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1/3 \\ 0 & -3 & -6 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

resulting in the form

$$\xrightarrow{-1/3R_2} \begin{bmatrix} 1 & 0 & -1 & -1/3 \\ 0 & 1 & 2 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 - x_3 = -1/3$, $x_2 + 2x_3 = 2/3$, or equivalently,

$$x_1 = x_3 - \frac{1}{3}, \qquad x_2 = -2x_3 + \frac{2}{3}$$

which implies that as the translation of a vector space the solutions are given by

$$\left\{ x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 2/3 \\ 0 \end{bmatrix} \right\}$$

2. (a) What is an elementary matrix?
(b) Express A as a product of elementary matrices where

$$A = \begin{bmatrix} 4 & -8 & 4 \\ -4 & 9 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

Solution:
(a) An elementary matrix is a row operation applied to the identity matrix:
(b) Solution **follows from**

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A &= \begin{bmatrix} 1 & -2 & 0 \\ -4 & 9 & 0 \\ 4 & -8 & 4 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A &= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Specifically,

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3. Use the matrix U and its transpose to show that

$$U = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is an orthogonal matrix. Also, show that the columns form an orthonormal basis for $\mathbb{R}^3$. (**B8, D4, D21**).

Solution: Since

$$U^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

we have that

$$UU^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likewise, $U^T U = I$, thus implying that U is orthogonal. Similarly, the columns of U are

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ \frac{2}{3} \end{bmatrix}$$

and $\|\mathbf{u}_1\|^2 = \|\mathbf{u}_2\|^2 = \|\mathbf{u}_3\|^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$. Also,

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{1}{3}\left(\frac{2}{3}\right) + \frac{2}{3}\left(\frac{-2}{3}\right) + \frac{2}{3}\left(\frac{1}{3}\right) = 0 \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= \frac{1}{3}\left(\frac{-2}{3}\right) + \frac{2}{3}\left(\frac{-1}{3}\right) + \frac{2}{3}\left(\frac{2}{3}\right) = 0 \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= \frac{2}{3}\left(\frac{-2}{3}\right) - \frac{2}{3}\left(\frac{-1}{3}\right) + \frac{1}{3}\left(\frac{2}{3}\right) = 0 \end{aligned}$$

4. State the definition of vector space. (**C1**)

5. Transform the basis $\{\langle 1,0,0,0\rangle, \langle 1,1,0,0\rangle, \langle 1,0,0,1\rangle, \langle 1,1,1,1\rangle\}$ for $\mathbb{R}^4$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

Solution: Let $\mathbf{w}_1 = \langle 1,0,0,0\rangle$. Then

$$\begin{aligned}
\mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 \\
&= \mathbf{v}_2 - \frac{1}{1}\mathbf{w}_1 \\
&= \langle 1,1,0,0\rangle - \langle 1,0,0,0\rangle \\
&= \langle 0,1,0,0\rangle
\end{aligned}$$

Likewise,

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\,\mathbf{w}_2 \\
&= \mathbf{v}_3 - \frac{1}{1}\mathbf{w}_1 - \frac{0}{1}\mathbf{w}_2 \\
&= \langle 1,0,0,1\rangle - \langle 1,0,0,0\rangle \\
&= \langle 0,0,0,1\rangle
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathbf{w}_3 &= \mathbf{v}_4 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\,\mathbf{w}_2 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_3}{\mathbf{w}_3 \cdot \mathbf{w}_3}\,\mathbf{w}_3 \\
&= \mathbf{v}_4 - \frac{1}{1}\mathbf{w}_1 - \frac{1}{1}\,\mathbf{w}_2 - \frac{1}{1}\,\mathbf{w}_3 \\
&= \langle 1,1,1,1\rangle - \langle 1,0,0,0\rangle - \langle 0,1,0,0\rangle - \langle 0,0,0,1\rangle \\
&= \langle 0,0,1,0\rangle
\end{aligned}$$

6. Prove the following: If A and B are $n \times n$ lower triangular matrices, then AB is also lower triangular. (**D1, D2, D11**)

Solution: Let $A = [a_{ij}]$ and $B = [b_{ij}]$, $i, j = 1, \ldots, n$. Then A and B lower triangular means that

$$a_{ij} = b_{ij} = 0 \ \ if \ \ j > i$$

Now suppose that $AB = [c_{ij}]$, $i, j = 1, \ldots, n$Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

and if $j > i$, then

$$c_{ij} = \sum_{k \le i} a_{ik} b_{kj} + \sum_{k > i} a_{ik} b_{kj}$$

If $k \le i$, however, then $k < j$, which implies that $b_{kj} = 0$. Hence, the first sum is 0. If $k > i$, then $a_{ik} = 0$. Hence, the second sum is also zero, thus showing that if $j > i$, then $c_{ij} = 0$.

7. Let V denote the space of all functions of the form

$$p(x) = ae^x + b + ce^{-x}$$

(that is, $V = \{ae^x + b + ce^{-x} \ : \ a, b, c \in \mathbb{R}\}$). Define a transformation by

$$(Tp)(x) = p(-x)$$

Show that T is a linear transformation on V, and find the matrix A which represents T relative to the basis $\{e^x, 1, e^{-x}\}$. (**C7, C9, C10**).

Solution: To find A, we apply T to the basis to obtain

$$\begin{aligned} T(e^x) &= e^{-x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} e^x & 1 & e^{-x} \end{bmatrix} \\ T(1) &= 1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} e^x & 1 & e^{-x} \end{bmatrix} \\ T\left(e^{-x}\right) &= e^x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} e^x & 1 & e^{-x} \end{bmatrix} \end{aligned}$$

Matrix A is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

8. Find the characteristic polynomial and eigenvalues of (**D14, D17, D18, D19**):

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

What is the dimension of each eigenspace? Explain.

Solution: To begin with,

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 & 4 \\ 2 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 2 & -\lambda \end{bmatrix}$$

and the cofactor definition of the determinant implies that

$$\begin{aligned} \chi(\lambda) = \det(A - \lambda I) &= -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 2 & -\lambda \end{vmatrix} - 4 \left| \begin{bmatrix} 2 & -\lambda & 0 \\ 0 & 1 & -\lambda \\ 0 & 0 & 2 \end{bmatrix} \right| \\ &= -\lambda\left(-\lambda^3\right) - 4(2 \cdot 1 \cdot 2) \\ &= \lambda^4 - 16 \end{aligned}$$

Since $\chi(\lambda) = \lambda^4 - 16$, the eigenvalues of A are

$$A = 2, 2i, -2, -2i$$

Each eigenvalue has multiplicity 1, and correspongingly, each eigenspace has dimension 1.

9. Find and describe the Eigenspaces of the matrix (**A9, D14, D17, D18, D19, D23**)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution: The eigenvalues are on the diagonal. For $\lambda = 1$, we have $(A - 1I)\,\mathbf{v} = 0$ requires the row reduction of

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right]$$

which yields

$$\xrightarrow{1/2R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which yields $2x_1 + x_2 = 0$. For $x_1 = x_2 = 0$, we must have $x_3 \neq 0$. Otherwise, we have

$$x_1 = \frac{-1}{2} x_2$$

so if $x_2 = 1$, then $x_1 = \frac{-1}{2}$. Thus, for $\lambda = 1$, we have

$$\textit{eigenvectors} = \textbf{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$$

For $\lambda = 2$, the system $(A - 2I)\,\mathbf{v} = 0$ requires the row reduction of

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 4 & 2 & -1 & 0 \end{array}\right]$$

which yields

$$\xrightarrow[4R_1 + R_3]{2R_1 + R_2} \left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right]$$

Thus, $-x_1 = 0$ and $2x_2 - x_3 = 0$, which implies that $x_3 = 2x_2$. Thus, the eigenvectors of A for eigenvalue $\lambda = 2$ are

$$\begin{bmatrix} 0 \\ x_2 \\ 2x_2 \end{bmatrix}, \quad a \in \mathbb{R}$$

10. Let A and B be $n \times n$ matrices over $\mathbb{R}$ and suppose that the eigenvectors of A form a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ over $\mathbb{R}^n$. Prove that if each $\mathbf{v}_j$, $j = 1, \ldots, n$ is also an eigenvector of B, then $AB = BA$. (**D17, D19, D23**)

Solution: Because the eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ over $\mathbb{R}^n$ form a linearly independent set, both A and B are diagonalizable. Moreover, they are diagonalized by the same transition matrix $P = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$, which is to say that there exists diagonal matrices D_A and D_B such that

$$A = PD_AP^{-1} \quad \textit{and} \quad B = PD_BP^{-1}$$

Thus, $AB = PD_AP^{-1}PD_BP^{-1} = PD_AD_BP^{-1}$ and likewise,

$$BA = PD_BP^{-1}PD_AP^{-1} = PD_BD_AP^{-1}$$

Since $D_AD_B = D_BD_A$ (multiplication of diagonal matrices is commutative), we must also have that $AB = BA$.