LINEAR ALGEBRA COMPREHENSIVE EXAM

<Semester,Yr> Prepared by <Faculty Member>

Exam Date

NAME ______________________________ STUDENT NUMBER ____________________

Be clear and **give all details**. Use all symbols correctly (such as equal signs). The bold faced numbers in parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!!!** You may omit two numbered problems. Indicate which two problems you are omitting: ______ and ______.

1. Find the solution set of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} \in \mathbb{R}^3$,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   and express the solution as the translation of a vector space. (**A3, A4, A5, D6**)

2. (a) What is an elementary matrix?
   (b) Express A as a product of elementary matrices where

$$A = \begin{bmatrix} 4 & -8 & 4 \\ -4 & 9 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

3. Use the matrix U and its transpose to show that

$$U = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{-1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

   is an orthogonal matrix. Also, show that the columns form an orthonormal basis for $\mathbb{R}^3$. (**B8, D4, D21**).

4. State the definition of vector space. (**C1**)

5. Transform the basis $\{\langle 1,0,0,0\rangle, \langle 1,1,0,0\rangle, \langle 1,0,0,1\rangle, \langle 1,1,1,1\rangle\}$ for $\mathbb{R}^4$ into an orthogonal basis using the Gram-Schmidt process. (**C17, C19, C20, C21**)

6. Prove the following: If A and B are $n \times n$ lower triangular matrices, then AB is also lower triangular. (**D1, D2, D11**)

7. Let V denote the space of all functions of the form

$$p(x) = ae^x + b + ce^{-x}$$

(that is, $V = \{ae^x + b + ce^{-x} \ : \ a, b, c \in \mathbb{R}\}$). Define a transformation by

$$(Tp)(x) = p(-x)$$

Show that T is a linear transformation on V, and find the matrix A which represents T relative to the basis $\{e^x, 1, e^{-x}\}$. (**C7, C9, C10**).

8. Find the characteristic polynomial and eigenvalues of (**D14, D17, D18, D19**):

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

What is the dimension of each eigenspace? Explain.

9. Find and describe the Eigenspaces of the matrix (**A9, D14, D17, D18, D19, D23**)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

10. Let A and B be $n \times n$ matrices over $\mathbb{R}$ and suppose that the eigenvectors of A form a linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ over $\mathbb{R}^n$. Prove that if each $\mathbf{v}_j$, $j = 1, \ldots, n$ is also an eigenvector of B, then $AB = BA$. (**D17, D19, D23**)