

CALCULUS COMPREHENSIVE EXAM
<Semester, Year>, Prepared by <Faculty Member>
Exam Date

NAME _____ STUDENT NUMBER _____

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: ____ and _____. There is a three hour time limit.

1. (a) State the definition of the limit of a function (that is, what does $\lim_{x \rightarrow a} f(x) = L$ mean?).
(b) Prove that if $\lim_{x \rightarrow a} f(x) = L$ and $k \neq 0$ is a constant, then $\lim_{x \rightarrow a} (kf(x)) = kL$ (**1, 2**)

Solution(a): $\lim_{x \rightarrow a} f(x) = L$ means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

(b) For all $\varepsilon > 0$, choose δ such that $|f(x) - L| < \varepsilon/|k|$. Then

$$\begin{aligned} |k| |f(x) - L| &< \varepsilon \\ |kf(x) - kL| &< \varepsilon \end{aligned}$$

which shows that $\lim_{x \rightarrow a} (kf(x)) = kL$.

2. (a) State the definition of a function $f(x)$ being continuous at $x = p$.
(b) Show that the following function is continuous at $x = 0$. (**4, 33**)

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Solution: (a) A function $f(x)$ is continuous at $x = p$ if (i) $f(p)$ exists, (ii) $\lim_{x \rightarrow p^+} f(x) = \lim_{x \rightarrow p^-} f(x)$, and

$$(iii) \lim_{x \rightarrow p} f(x) = f(p)$$

3. Prove that if f is differentiable at $x = p$, then f is also continuous at $x = p$. (**4,7**)

Solution: If $f(x)$ is differentiable at p , then

$$\begin{aligned}\lim_{x \rightarrow p} (f(x) - f(p)) &= \lim_{x \rightarrow p} (x - p) \left(\frac{f(x) - f(p)}{x - p} \right) \\ &= \lim_{x \rightarrow p} (x - p) \cdot \lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p} \\ &= 0 \cdot f'(p) \\ &= 0\end{aligned}$$

However, $\lim_{x \rightarrow p} (f(x) - f(p)) = 0$ is equivalent to

$$\lim_{x \rightarrow p} f(x) = f(p)$$

which shows that $f(x)$ is continuous at p .

4. Consider the function

$$f(x) = \frac{x^4}{1 - x^4}$$

Find the first and second derivative of f , determine where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f , find the extrema of f , and graph $y = f(x)$ (**3, 8, 14, 15, 16, 17**)

Solution: To begin with, f has vertical asymptotes at $x = 1$ and $x = -1$, and it has a horizontal asymptote of $y = -1$. Next,

$$\begin{aligned}f'(x) &= \frac{d}{dx} \frac{x^4}{1 - x^4} = \frac{4x^3}{(1 - x^4)^2} \\ f''(x) &= \frac{d}{dx} \frac{4x^3}{(1 - x^4)^2} = \frac{12x^2 + 20x^6}{(1 - x^4)^3}\end{aligned}$$

The sign of $f'(x)$ depends on the sign of $4x^3$, so

f increasing on $(0, 1) \cup (1, \infty)$

f decreasing on $(-\infty, -1) \cup (-1, 0)$

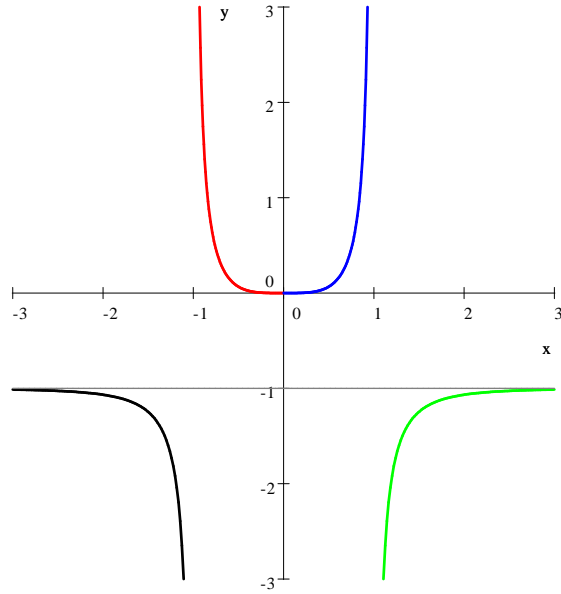
Thus, $f(x)$ has a relative minimum of $f(0) = 0$ at $x = 0$.

Similarly, the sign of $f''(x)$ depends on $1 - x^4$, so

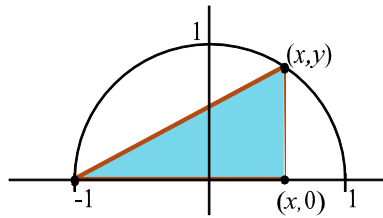
f concave up on $(-1, 0) \cup (0, 1)$

f concave down on $(-\infty, 1) \cup (1, \infty)$

The graph of $y = f(x)$ is thus as shown below:



5. A triangle has vertices at $(-1, 0)$, $(x, 0)$, and (x, y) , where $x^2 + y^2 = 1$ and $y \geq 0$ (see below). For what point (x, y) on the unit circle is the area of the triangle a maximum? **(18)**



Solution: The base of the triangle has length $x + 1$, while the height $y = \sqrt{1 - x^2}$. Thus, the area is

$$A = \frac{1}{2}(x + 1)\sqrt{1 - x^2}$$

Clearly, the area is 0 at $x = 1$ or $x = -1$, so we need to find the critical point $p \in [-1, 1]$ for which $A'(p) = 0$. However,

$$\begin{aligned} A'(x) &= \frac{d}{dx} \left(\frac{1}{2}(x + 1)\sqrt{1 - x^2} \right) \\ &= \frac{1 - x^2}{2\sqrt{1 - x^2}} - \frac{x + x^2}{2\sqrt{1 - x^2}} \\ &= \frac{1 - x - 2x^2}{2\sqrt{1 - x^2}} \\ &= \frac{(x + 1)(1 - 2x)}{2\sqrt{1 - x^2}} \end{aligned}$$

and clearly, $A'(x) = 0$ if $x = 1/2$. Since $A(1/2) = \frac{3\sqrt{3}}{8} > 0$, the maximum area of the triangle occurs when $x = \frac{1}{2}$.

6. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x) dx$.
 (b) State the Fundamental Theorem of Calculus (both parts) (**20, 21, 23**)

Solution: A partition of an interval $[a, b]$ is a collection of subintervals $[x_{k-1}, x_k]$, $k = 1, \dots, n$ where

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$

The norm of a partition $P = \{[x_{k-1}, x_k]\}_{k=1}^n$ is defined

$$\|P\| = \max_{k=1, \dots, n} \{\Delta x_k\}$$

where $\Delta x_k = x_k - x_{k-1}$, $k = 1, \dots, n$. A Riemann sum for a function $f(x)$ defined on $[a, b]$ is defined

$$\sum_{k=1}^n f(t_k) \Delta x_k$$

where $P = \{[x_{k-1}, x_k]\}_{k=1}^n$ is a partition of $[a, b]$ and $t_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$ for all $k = 1, \dots, n$. If for $f(x)$ defined on $[a, b]$, if there exists a number S such that

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(t_k) \Delta x_k = S$$

where the limit is over all partitions of $[a, b]$, then S is called the *definite integral* of $f(x)$ over $[a, b]$ and is denoted

$$S = \int_a^b f(x) dx$$

(b) **Part 1:** If $f(x)$ is continuous on $[a, b]$ and we define $F(x)$ by

$$F(x) = \int_0^x f(t) dt$$

then $F'(x) = f(x)$ for all $x \in (a, b)$. **Part 2:** If $f(x)$ is continuous on $[a, b]$ and $G(x)$ is *any* antiderivative of $f(x)$ over $[a, b]$, then

$$\int_a^b f(x) dx = G(b) - G(a)$$

7. (a) Use the Fundamental Theorem of Calculus to find the derivative $F'(x)$ of

$$F(x) = \int_0^{\ln(x)} \frac{dt}{e^t + 1}$$

Simplify completely. **(8, 23)**

(b) Use the Fundamental Theorem of Calculus to evaluate

$$\int_0^{\pi/2} \frac{\cos(x)}{2 - \sin(x)} dx$$

and indicate with a star (*) where you are applying the Fundamental Theorem. **(24, 35)**

Solution: (a) The function $F(x)$ is of the form

$$F(x) = G(u(x)) \quad \text{where } G(u) = \int_0^u \frac{dt}{e^t + 1} \quad \text{and } u = \ln(x)$$

Thus, $F'(x) = G'(u(x)) u'(x)$, so that

$$\begin{aligned} F'(x) &= \frac{1}{e^u + 1} \frac{du}{dx} \quad \text{where } u = \ln(x) \\ &= \frac{1}{e^{\ln(x)} + 1} \frac{1}{x} \\ &= \frac{1}{x(x+1)} \end{aligned}$$

(b) Let $u = 2 - \sin(x)$, $du = -\cos(x) dx$. Then

$$\int \frac{\cos(x) dx}{2 - \sin(x)} = \int \frac{-1}{u} du = -\ln(u) + C = -\ln(2 - \sin(x)) + C$$

Thus, $-\ln(2 - \sin x)$ is an antiderivative of $\frac{\cos(x)}{2 - \sin(x)}$ on $[0, \frac{\pi}{2}]$, so that

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos(x)}{2 - \sin(x)} dx & \overset{*}{=} -\ln\left(2 - \sin\frac{\pi}{2}\right) - (-\ln(2 - \sin 0)) \\ &= -\ln(2 - 1) + \ln(2 - 0) \\ &= -\ln(1) + \ln(2) \\ &= \ln(2) \end{aligned}$$

8. What is the length of the curve $y = \cosh(x)$ for x in $[0, \ln(2)]$? **(23, 24, 27)**

Solution: Since $y' = \sinh(x)$, the arclength integral is

$$\begin{aligned}
 L &= \int_0^{\ln(2)} \sqrt{1 + (y')^2} dx \\
 &= \int_0^{\ln(2)} \sqrt{1 + \sinh^2(x)} dx \\
 &= \int_0^{\ln(2)} \sqrt{\cosh^2(x)} dx \\
 &= \int_0^{\ln(2)} \cosh(x) dx \\
 &= \sinh(\ln 2) - \sinh(0) \\
 &= \frac{3}{4}
 \end{aligned}$$

9. Do each of the following.

(a) Evaluate $\lim_{x \rightarrow 0^+} (1-x)^{1/x}$. **(31, 37)**

(b) Evaluate $\int_1^{\infty} \frac{dx}{x+1}$. **(24, 38)**

(c) Evaluate $\int_0^1 x \ln(x) dx$. **(39)**

Solution: (a) Let $L = \lim_{x \rightarrow 0^+} \ln(1-x)^{1/x}$. Then

$$L = \lim_{x \rightarrow 0^+} \frac{\ln(1-x)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{-1}{1-x}}{1} = -1$$

Since e^x is continuous,

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} (1-x)^{1/x} &= \lim_{x \rightarrow 0^+} e^{\ln(1-x)^{1/x}} \\
 &= e^{\lim_{x \rightarrow 0^+} \frac{\ln(1-x)}{x}} \\
 &= e^{-1}
 \end{aligned}$$

(b) We write the integral as an improper integral in the form

$$\lim_{R \rightarrow \infty} \int_0^R \frac{dx}{x+1} = \lim_{R \rightarrow \infty} \ln(R+1) = \infty$$

Thus, the integral diverges.

(c) We write the integral as an improper integral in the form

$$\int_0^1 x \ln(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x \ln(x) dx$$

Integration by parts with $u = \ln(x)$ and $dv = x dx$ leads to

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x \ln(x) dx &= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{x^2}{2} \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 \frac{x^2}{2} \frac{1}{x} dx \right) \\
 &= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{2} \ln(1) - \frac{\varepsilon^2}{2} \ln(\varepsilon) - \int_{\varepsilon}^1 \frac{x}{2} dx \right) \\
 &= -\frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\ln(\varepsilon)}{\varepsilon^{-2}} + \int_{\varepsilon}^1 x dx \right) \\
 &= -\frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1/\varepsilon}{-2\varepsilon^{-3}} + \frac{1}{2} - \frac{\varepsilon^2}{2} \right) \\
 &= -\frac{1}{2} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\varepsilon^2}{-2} + \frac{1}{2} - \frac{\varepsilon^2}{2} \right) \\
 &= -\frac{1}{4}
 \end{aligned}$$

10. (a) State the definition of the limit of a sequence: $\lim_{n \rightarrow \infty} a_n = L$. **(41, 43)**

(b) State the definition of the sum of a series: $\sum_{n=1}^{\infty} a_n = S$.

(c) Determine if the following series converges absolutely, converges conditionally, or diverges

$$\sum_{n=1}^{\infty} \frac{(-1)^n + n}{n^3}$$

You may use any test, but you must show that the hypotheses of the test you use are satisfied.

Solution: (a) $\lim_{n \rightarrow \infty} a_n = L$ means that for all $\varepsilon > 0$, there exists $N > 0$ such that if $n > N$, then $|a_n - L| < \varepsilon$

(b) $\sum_{n=1}^{\infty} a_n = S$ means that for all $\varepsilon > 0$, there exists $M > 0$ such that if $N > M$, then

$$\left| S - \sum_{n=1}^N a_n \right| < \varepsilon$$

(c) Since $(-1)^n + n < 2n$ for all $n \in \mathbb{Z}^+$, we can compare to $\sum \frac{2n}{n^3} = 2 \sum \frac{1}{n^2}$. That is,

$$S_N = \sum_{n=1}^N \frac{(-1)^n + n}{n^3}$$

is an increasing sequence and

$$S_N \leq \sum_{n=1}^N \frac{2n}{n^3} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Thus, $\{S_N\}$ is also bounded, which implies that $\{S_N\}$ converges, implying that the series converges absolutely (all terms are non-negative).

10. Do each of the following (46):

(a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n (x - a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e., on what types of sets might the series converge conditionally, converge absolutely, or diverge).

(b) What is the radius of converge of the series

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$$

(give detailed reasons for your answer).

Solution: (a) There exists $R \geq 0$ (and possibly $R = \infty$) such that

$$\begin{aligned} \sum_{n=0}^{\infty} c_n (x - a)^n & \text{ converges absolutely on } (a - R, a + R) \\ \sum_{n=0}^{\infty} c_n (x - a)^n & \text{ diverges on } (-\infty, a - R) \cup (a + R, \infty) \end{aligned}$$

At $x = a - R$ and $x = a + R$, the series may converge conditionally, converge absolutely, or diverge.

(b) The ratio test with $a_n = \frac{n+1}{n!} x^n$ implies that

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+2)/(n+1)! x^{n+1}}{(n+1)/n! x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{n+2}{n+1} \frac{n!}{(n+1)!} x \right| \\ &= |x| \left(\lim_{n \rightarrow \infty} \frac{n+2}{n+1} \right) \left(\lim_{n \rightarrow \infty} \frac{n!}{(n+1)n!} \right) \\ &= |x| \left(\lim_{n \rightarrow \infty} \frac{1}{n+1} \right) \\ &= 0 \end{aligned}$$

Thus, the radius of convergence is $R = \infty$, which means that the series converges absolutely everywhere.

11. Do each of the following (44, 47):

(a) Use the MacLaurin Series for $f(x) = \sin(x)$ to find the Maclaurin series expansion of $\int_0^x \sin(u^4) du$.

(b) Use the series in (a) to calculate the limit

$$\lim_{x \rightarrow 0} \frac{1}{x^5} \int_0^x \sin(u^4) du$$

Solution: (a) Since $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, it follows that

$$\sin(u^4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (u^4)^{2n+1}$$

and correspondingly,

$$\begin{aligned} \int_0^x \sin(u^4) du &= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (u^4)^{2n+1} du \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x u^{8n+4} du \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{8n+5}}{8n+5} \end{aligned}$$

(b) The result in (a) can be written

$$\begin{aligned} \int_0^x \sin(u^4) du &= \frac{x^5}{5} - \frac{x^{13}}{3!(13)} + \dots + \frac{(-1)^n x^{8n+5}}{(2n+1)!(8n+5)} \\ &= \frac{x^5}{5} + x^5 g(x) \end{aligned}$$

where $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{8n}}{(2n+1)!(8n+5)}$ is continuous and $g(0) = 0$. Thus,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{x^5} \int_0^x \sin(u^4) du &= \lim_{x \rightarrow 0} \frac{1}{x^5} \left(\frac{x^5}{5} + x^5 g(x) \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{1}{5} + g(x) \right) \\ &= \frac{1}{5} \end{aligned}$$