CALCULUS COMPREHENSIVE EXAM

<Semester, Year>, Prepared by <Faculty Member>

Exam Date

NAME _____ STUDENT NUMBER _____

Be clear and **give all details**. Use symbols correctly (such as equal signs). The numbers in bold faced parentheses indicate the number of the topics covered in that problem from the Study Guide. **No calculators!** You may omit one problem from numbers 1 through 6 (which contain Calculus 1 material) and one problem from numbers 7 through 12 (which contain Calculus 2 material). Indicate which two problems you are omitting: _____ and _____. There is a three hour time limit.

1. (a) State the definition of the limit of a function (that is, what does $\lim_{x\to a} f(x) = L$ mean?).
   (b) Prove that if $\lim_{x\to a} f(x) = L$ and $k \neq 0$ is a constant, then $\lim_{x\to a}(kf(x)) = kL$ **(1, 2)**

2. Prove that if f is differentiable at $x = p$, then f is also continuous at $x = p$. (**4,7**)

3. (a) State the definition of a function $f(x)$ being continuous at $x = p$.
   (b) Show that the following function is continuous at $x = 0$. (**4, 33)**

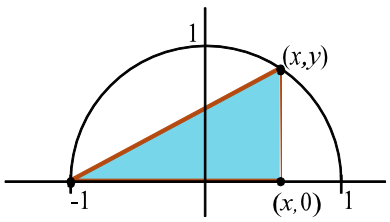
$$f(x) = \begin{cases} e^{-1/x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

4. Consider the function

$$f(x) = \frac{x^4}{1 - x^4}$$

   Find the first and second derivative of f, determine where f is increasing/decreasing, find where f is concave up/concave down, find the asymptotes of the graph of f, find the extrema of f, and graph $y = f(x)$ **(3, 8, 14, 15, 16, 17)**

5. A triangle has vertices at $(-1, 0)$, $(x, 0)$, and (x, y), where $x^2 + y^2 = 1$ and $y \geq 0$ (see below). For what point (x, y) on the unit circle is the area of the triangle a maximum? **(18)**



6. (a) State the definition of *partition*, *norm* of a partition, *Riemann sum*, and *definite integral* for $\int_a^b f(x)\,dx$.
   (b) State the Fundamental Theorem of Calculus (both parts) (**20, 21, 23**)

7. **(a)** Use the Fundamental Theorem of Calculus to find the derivative $F'(x)$ of
$$F(x) = \int_0^{\ln(x)} \frac{dt}{e^t + 1}$$
Simplify completely. **(8, 23)**
   (b) Use the Fundamental Theorem of Calculus to evaluate
$$\int_0^{\pi/2} \frac{\cos(x)}{2 - \sin(x)}\,dx$$
and indicate with a star $(*)$ where you are applying the Fundamental Theorem. **(24, 35)**

8. What is the length of the curve $y = \cosh(x)$ for x in $[0, \ln(2)]$? **(23, 24, 27)**

9. Do each of the following.
   (a) Evaluate $\lim_{x\to 0^+} (1-x)^{1/x}$. **(31, 37)**
   (b) Evaluate $\int_1^\infty \frac{dx}{x+1}$. **(24, 38)**
   (c) Evaluate $\int_0^1 x \ln(x)\,dx$. **(39)**

10. **(a)** State the definition of the limit of a sequence: $\lim_{n\to\infty} a_n = L$. **(41, 43)**
    (b) State the definition of the sum of a series: $\sum_{n=1}^{\infty} a_n = S$.
    (c) Determine if the following series converges absolutely, converges conditionally, or diverges
$$\sum_{n=1}^{\infty} \frac{(-1)^n + n}{n^3}$$
You may use any test, but you must show that the hypotheses of the test you use are satisfied.

10. Do each of the following (**46**)**:**
    (a) For a given x value, the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ may converge conditionally, converge absolutely, or diverge. Describe the possible behavior of this series (i.e., on what types of sets might the series converge conditionally, converge absolutely, or diverge).
    (b) What is the radius of converge of the series
$$\sum_{n=0}^{\infty} \frac{n+1}{n!} x^n$$
(give detailed reasons for your answer).

11. Do each of the following (**44, 47**):

(a) Use the MacLaurin Series for $f(x) = \sin(x)$ to find the Maclaurin series expansion of $\int_0^x \sin(u^4)\ du$.

(b) Use the series in (a) to calculate the limit

$$\lim_{x \to 0} \frac{1}{x^5} \int_0^x \sin\left(u^4\right) du$$